PART - B

Unit - VI

Many problems in all branches of science and engineering when analysed for putting in a mathematical form assumes the form of a *differential equation*. An engineer or an applied mathematician will be mostly intrested in obtaining a solution for the associated equation without bothering much on the rigorous aspects like the proof, validity conditions, region of existence etc. Accordingly the study of differential equations at various levels is focussed on the methods of solving the equations.

If $y = f(x)$ is an unknown function, an equation which involves atleast one derivative of y *w.r.t* x is called an *ordinary differential equation* which in future will be simply referred to as a *Differential Equation (D.E)*.

The *order* of the D.E is the order of the highest derivative present in the equation and the *degree* of the D.E is the degree of the highest order derivative after clearing the fractional powers.

Finding y as a function of x explicitly $[y = f(x)]$ or a relationship in x and y satisfying the D . E $[f(x, y) = c]$ constitutes the solution of the D.E.

Observe the following *examples* along with their order and degree.

1. $\frac{dy}{dx} = 2x$ [order = 1, degree = 1]

2. $\left(\frac{dy}{dx}\right)^2 + 3\left(\frac{dy}{dx}\right) + 2 = 0$ [order = 1, degree = 2]

3. $\frac{d^2 x}{dt^2} + w^2 x = 0$ [order = 2, degree = 1]

4. $\frac{d^3 y}{dx^3} + 5\left(\frac{d^2 y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^3 = \sin x$ [order = 3, degree = 1]

5. $\left(r^2+r_1^2\right)^{3/2} = r^2+2r_1^2-r\,r_2$ where $r_1 = \dfrac{dr}{d\theta},\quad r_2 = \dfrac{d^2r}{d\theta^2}$

Here we need to clear the fractional power 3/2 by squaring both sides.

$$\therefore \quad (r^2+r_1^2)^3 = \left(r^2+2r_1^2-r\,r_2\right)^2$$

Observing the term r_2^2 in the R.H.S we conclude that the order of the equation is 2 and the degree is 2.

General solution and particular solution

A solution of a D.E is a relation between the dependent and independent variables satisfying the given equation identically.

For example $y = e^{3x}$ is a solution of the D.E

$$\frac{dy}{dx}-3e^{3x} = 0 \text{ as we could see that}$$

$$\frac{d}{dx}(e^{3x})-3e^{3x} = 3e^{3x}-3e^{3x} = 0$$

Also $x = 2\cos wt+3\sin wt$ is a solution of the D.E

$$\frac{d^2x}{dt^2} + w^2x = 0 \text{ as we have,}$$

$$\frac{d^2}{dt^2}\left(2\cos wt+3\sin wt\right)+w^2\left(2\cos wt+3\sin wt\right)$$

$$= -2w^2\cos wt-3w^2\sin wt+2w^2\cos wt+3w^2\sin wt = 0$$

Now let us look at the first of the above example in the reverse way.
That is to consider the D.E

$$\frac{dy}{dx} - 3e^{3x} = 0 \text{ and try to find } y.$$

i.e., $\dfrac{dy}{dx} = 3e^{3x}$ and to find y we need to get rid off $\dfrac{d}{dx}$ being the differential operator. Obviously we have to use anti differentiation, that is integration.

$$\frac{dy}{dx} = 3e^{3x}$$

$$\Rightarrow \quad y = \int 3e^{3x}\,dx+c \quad ie., \quad y = 3\cdot\frac{e^{3x}}{3}+c$$

or $\quad y = e^{3x}+c,$ c being the constant of integration.

Integration is directly or indirectly involved in the process of getting a solution of the given D.E and accordingly the solution will be involved with arbitrary constants. Such a solution is called as the **general solution** of the D.E. It is obvious that the number of arbitrary constants present in the solution is equal to the order of the D.E.

In the first example we verified that e^{3x} is a solution of the equation $y' - 3e^{3x} = 0$. We obtained $y = e^{3x} + c$ as the solution by solving the equation, c being arbitrary. The solution $y = e^{3x}$ is a particular case of the solution $e^{3x} + c$ (*for* $c = 0$) and accordingly it is called a particular solution and the same can be interpreted as follows.

If the arbitrary constants present in the solution are evaluated by using a set of given conditions then the solution so obtained is called a **particular solution**. In many physical problems these conditions can be formulated from the problem itself.

In the discussion of the first example, let us consider the initial condition $y(0) = 1$ That is $y = 1$ when $x = 0$.

The general solution $y = e^{3x} + c$ becomes

$$1 = e^0 + c \text{ or } 1 = 1 + c \text{ which gives } c = 0.$$

Thus $y = e^{3x}$ is a particular solution of the D.E. In this context we can also say that the solution of an O.D.E is unique in the sense that, the solution can differ only by constants. It may be just noted that the general solution of the D.E

$$\frac{d^2x}{dt^2} + w^2 x = 0 \text{ is } x = c_1 \cos wt + c_2 \sin wt$$

Suppose we impose the conditions $x(0) = 2$ and $x'(0) = 3w$ which means that $x = 2$, $\frac{dx}{dt} = 3w$ when $t = 0$, then we obtain $c_1 = 2$ and $c_2 = 3$.

It is equivalent to saying that $x = 2\cos wt + 3\sin wt$ is a particular solution of the D.E : $\frac{d^2x}{dt^2} + w^2 x = 0$.

In this chapter we first discuss various methods of solving differential equations of first order and first degree.

Note : *Basic integration and integration methods are* ***essential prerequisites*** *for this chapter.*

Recollecting the definition of the order and the degree of a D.E, a first order and first degree equation will be of the form

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad M(x, y)\,dx + N(x, y)\,dy = 0.$$

The mainly classified four types of differential equations of first order and first degree are as follows :

1. Variables separable equations
2. Homogeneous equations
3. Exact equations
4. Linear equations

If the given D.E can be put in the form such that the coefficient of dx is a function of the variable x only and the coefficient of dy is a function of y only then the given equation is said to be in the separable form.
The modified form of such an equation will be,

$P(x)\,dx + Q(y)\,dy = 0.$ Integrating we have

$$\int P(x)\,dx + \int Q(y)\,dy = c$$

This is the general solution of the equation.

>> $\sec^2 x \tan y \; dx + \sec^2 y \tan x \; dy = 0$

Dividing throughout by $\tan y \cdot \tan x$ we get

$$\frac{\sec^2 x}{\tan x}dx + \frac{\sec^2 y}{\tan y}dy = 0 \quad (\textit{Variables are separated})$$

$$\Rightarrow \quad \int \frac{\sec^2 x}{\tan x}dx + \int \frac{\sec^2 y}{\tan y}dy = c$$

ie., $\log(\tan x) + \log(\tan y) = c$

ie., $\log(\tan x \cdot \tan y) = \log k$ *(say)*

Thus $\mathbf{\tan x \tan y = k}$ is the required solution.

>> $\dfrac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$

ie., $\dfrac{dy}{dx} = e^{-2y}\,(e^{3x} + x^2)$

or $\dfrac{dy}{e^{-2y}} = (e^{3x} + x^2)\,dx$ by separating the variables.

$\Rightarrow$ $\int e^{2y}\,dy - \int (e^{3x} + x^2)\,dx = c$

Thus $\dfrac{e^{2y}}{2} - \dfrac{e^{3x}}{3} - \dfrac{x^3}{3} = c$, is the required solution.

>> $x\,y\,\dfrac{dy}{dx} = 1 + x + y + x\,y$

ie., $x\,y\,\dfrac{dy}{dx} = (1 + x) + y\,(1 + x)$

ie., $x\,y\,\dfrac{dy}{dx} = (1 + x)(1 + y)$

or $\dfrac{y\,dy}{1+y} = \dfrac{1+x}{x}\,dx$ by separating the variables.

$\Rightarrow$ $\int \dfrac{y}{1+y}\,dy - \int \dfrac{1+x}{x}\,dx = c$

or $\int \dfrac{(1+y)-1}{1+y}\,dy - \int \dfrac{1}{x}\,dx - \int 1\,dx = c$

ie., $\int 1\,dy - \int \dfrac{1}{1+y}\,dy - \log x - x = c$

ie., $y - \log(1 + y) - \log x - x = c$

Thus $(y - x) - \log[\, x (1 + y) \,] = c$, is the required solution.

Note : *Some differential equations can be reduced to the variables separable form by taking a suitable substitution. We identify a few types of equation along with the associated substitutions.*

(i) $\frac{dy}{dx} = f(ax + by + c)$; substitution : $t = ax + by + c$

(ii) $\frac{dy}{dx} = \frac{(ax + by) + c}{k(ax + by) + c'}$; substitution : $t = ax + by$

Solve : $\frac{dy}{dx} = (9x + y + 1)^2$

>> We have $\frac{dy}{dx} = (9x + y + 1)^2$ . . . (1)

Put $t = 9x + y + 1 \quad \therefore \quad \frac{dt}{dx} = 9 + \frac{dy}{dx}$ or $\frac{dt}{dx} - 9 = \frac{dy}{dx}$

Now (1) becomes $\frac{dt}{dx} - 9 = t^2$ or $\frac{dt}{dx} = t^2 + 9$

Hence $\int \frac{dt}{t^2 + 3^2} = \int dx + c$ *ie.*, $\frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right) - x = c$

Thus $\mathbf{\frac{1}{3}\tan^{-1}\left(\frac{9x + y + 1}{3}\right) - x = c}$, is the required solution.

Solve : $\frac{dy}{dx} = \frac{x + y + 1}{2x + 2y + 3}$

>> We have $\frac{dy}{dx} = \frac{(x + y) + 1}{2(x + y) + 3}$

Put $t = x + y \quad \therefore \quad \frac{dt}{dx} = 1 + \frac{dy}{dx}$ or $\frac{dt}{dx} - 1 = \frac{dy}{dx}$

The given equation becomes,

$$\frac{dt}{dx} - 1 = \frac{t + 1}{2t + 3} \quad \text{or} \quad \frac{dt}{dx} = 1 + \frac{t + 1}{2t + 3}$$

ie., $\frac{dt}{dx} = \frac{2t + 3 + t + 1}{2t + 3}$ or $\frac{dt}{dx} = \frac{3t + 4}{2t + 3}$

ie., $\frac{2t + 3}{3t + 4}\,dt = dx$

$\Rightarrow \quad \int \frac{2t + 3}{3t + 4}\,dt - \int dx = c$ . . . (1)

Let $2t + 3 = l(3t + 4) + m$

or $\quad 2t+3 = (3l)t + (4l+m)$

$\Rightarrow \quad 3l = 2$ and $4l + m = 3$

$\therefore \quad l = 2/3$ and $8/3 + m = 3$ or $m = 1/3$.

Thus $\quad 2t+3 = 2/3 \cdot (3t+4) + 1/3$

Hence (1) becomes $\displaystyle\int \frac{2/3 \cdot (3t+4) + 1/3}{3t+4}\, dt - \int dx = c$

ie., $\quad \displaystyle\frac{2}{3}\int 1\, dt + \frac{1}{3}\int \frac{dt}{3t+4} - x = c$

ie., $\quad \displaystyle\frac{2t}{3} + \frac{1}{9}\log(3t+4) - x = c$, where $t = x + y$

$\therefore \quad \displaystyle\frac{2(x+y)}{3} + \frac{1}{9}\log(3x+3y+4) - x = c$

Thus $\quad \displaystyle\mathbf{\frac{1}{3}(2y-x) + \frac{1}{9}\log(3x+3y+4) = c}$, is the required solution.

[illegible]

Definition of homogeneous [illegible]

A function $u = f(x, y)$ is said to be a *homogeneous function of degree 'n'* if

$$u = x^n\, g(y/x) \text{ or } u = y^n\, g(x/y)$$

Examples

1. $\quad u = 2x + 3y$

   $u = x[2 + 3(y/x)] = x^1 g(y/x)$

$\Rightarrow \quad u$ is a homogeneous function of degree 1.

2. $\quad u = x^2 - xy + y^2$

   $u = x^2 [1 - (y/x) + (y/x)^2] = x^2 g(y/x)$

$\Rightarrow \quad u$ is a homogeneous function of degree 2.

3. $\quad u = xy^2 + x^2 y$

   $u = x^3 [(y/x)^2 + (y/x)] = x^3 g(y/x)$

$\Rightarrow \quad u$ is a homogeneous function of degree 3.

Note : *In these examples we observe that the total degree of the terms involved in each of the functions are respectively 1, 2, 3 and hence the homogeneous aspect can be recognized instantly.*

4. $u = x\cos(y/x) + y\sin(y/x)$

$u = x[\cos(y/x) + (y/x)\sin(y/x)] = x^1 g(y/x)$

$\Rightarrow$ u is homogeneous of degree 1.

5. $u = x^2 \tan(x/y) + y^2$

$u = y^2 [(x/y)^2 \tan(x/y) + 1] = y^2 g(x/y)$

$\Rightarrow$ u is homogeneous of degree 2.

6. $u = x\log y - x\log x$

$u = x(\log y - \log x) = x\log(y/x) = x^1 g(y/x)$

$\Rightarrow$ u is homogeneous of degree 1.

7. $u = y + \sqrt{xy}$

$$u = x\left[(y/x) + \frac{1}{x}\sqrt{xy}\right] = x\left[(y/x) + \sqrt{xy/x_2}\right] = x\left[(y/x) + \sqrt{y/x}\right] = x^1 g(y/x)$$

$\Rightarrow$ u is homogeneous of degree 1.

8. $u = \sqrt{x^2 + y^2}$

$u = \sqrt{x^2[1 + (y/x)^2]} = x\sqrt{1 + (y/x)^2} = x^1 g(y/x)$

$\Rightarrow$ u is homogeneous of degree 1.

Remark : *These examples gives an insight to recognize the homogeneous aspect of functions along with the associated degree instantly.*

[illegible]

A D.E of the form $M(x, y)\,dx + N(x, y)\,dy = 0$ is said to be a homogeneous d.e if both $M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree.

[illegible]

We prefer to have the differential equation in the form $\dfrac{dy}{dx} = \dfrac{f(x, y)}{g(x, y)}$
after recognizing that the D.E is a homogeneous one.
We take the substitution $y = vx$ so that,

$$\frac{dy}{dx} = v \cdot 1 + x\frac{dv}{dx}\text{, by product rule.}$$

[*Since both $f(x, y)$ and $g(x, y)$ are expressible in the form $x^n F(y/x)$ it will be feasible to take a substitution for y/x. Thus if $v = y/x$ then it is obvious that v is a function of x as y is a function of x. For convenience we take $y = vx$ and then differentiate w.r.t x*]

Substituting for y and $\frac{dy}{dx}$ the given equation after simplification converts into an equation which can be solved by separating the variables v and x.

Finally we substitute back for v which is equal to y/x.

If the coefficient of dx and dy involves terms with (x/y) then we have to write the equation associated with $\frac{dx}{dy}$ and use the substitution $x = vy$.

This gives $\frac{dx}{dy} = v \cdot 1 + y\frac{dv}{dy}$. We then proceed as described earlier.

(Observe that the coefficient of dx and dy are homogeneous functions of degree 2)

>> We have $\frac{dy}{dx} = \frac{x^2 - y^2}{xy}$ . . . (1)

Put $y = vx \quad \therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

Now (1) becomes, $v + x\frac{dv}{dx} = \frac{x^2 - v^2x^2}{x \cdot vx}$

ie., $v + x\frac{dv}{dx} = \frac{x^2(1 - v^2)}{x^2 v}$

ie., $x\frac{dv}{dx} = \frac{1 - v^2}{v} - v$ or $x\frac{dv}{dx} = \frac{1 - 2v^2}{v}$

$\therefore$ $\frac{v}{1 - 2v^2}\, dv = \frac{dx}{x}$, by separating the variables.

$\Rightarrow$ $\int \frac{v}{1 - 2v^2}\, dv - \int \frac{dx}{x} = c$

ie., $-\frac{1}{4}\log(1 - 2v^2) - \log x = c$

ie., $\log(1 - 2v^2) + 4\log x = -4c$

ie., $\log\left[(1 - 2v^2)x^4\right] = \log k$ *(say)* $\quad \because \quad 4\log x = \log x^4$

or $(1 - 2v^2)x^4 = k$ where $v = y/x$

ie., $\left(1 - \frac{2y^2}{x^2}\right)x^4 = k$ or $x^2(x^2 - 2y^2) = k$

Thus $\mathbf{x^4 - 2x^2y^2 = k}$, is the required solution.

(Observe that the coefficient of dx and dy are homogeneous functions of degree 3)

>> The given equation can be written as,

$$\frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3} \qquad \ldots (1)$$

Put $y = v x \quad \therefore \quad \frac{dy}{dx} = v + x \frac{dv}{dx}$

Now (1) becomes, $v + x \frac{dv}{dx} = \frac{x^2 \cdot vx}{x^3 + v^3 x^3}$

ie., $v + x \frac{dv}{dx} = \frac{x^3 v}{x^3 (1 + v^3)}$ or $x \frac{dv}{dx} = \frac{v}{1 + v^3} - v$

ie., $x \frac{dv}{dx} = \frac{v - v - v^4}{1 + v^3}$ or $x \frac{dv}{dx} = \frac{-v^4}{1 + v^3}$

$\therefore \quad \frac{1 + v^3}{v^4} dv = \frac{-dx}{x}$, by separating the variables.

$\Rightarrow \quad \int \frac{1}{v^4} dv + \int \frac{1}{v} dv + \int \frac{dx}{x} = c$

ie., $\frac{v^{-3}}{-3} + \log v + \log x = c$

ie., $-\frac{1}{3v^3} + \log (vx) = c$, where $v = \frac{y}{x}$

Thus $\frac{-x^3}{3y^3} + \log y = c$, is the required solution.

>> The given equation can be written as,

$$\frac{dy}{dx} = \frac{y^3 - 3x^2 y}{x^3 - 3x y^2} \qquad \ldots (1)$$

Put $y = vx \quad \therefore \quad \frac{dy}{dx} = v + x \frac{dv}{dx}$

Now (1) becomes, $v + x \frac{dv}{dx} = \frac{v^3 x^3 - 3x^2 \cdot vx}{x^3 - 3x \cdot v^2 x^2}$

ie., $v + x\dfrac{dv}{dx} = \dfrac{x^3(v^3-3v)}{x^3(1-3v^2)}$ or $x\dfrac{dv}{dx} = \dfrac{v^3-3v}{1-3v^2} - v$

ie., $x\dfrac{dv}{dx} = \dfrac{v^3-3v-v+3v^3}{1-3v^2}$

ie., $x\dfrac{dv}{dx} = \dfrac{4(v^3-v)}{1-3v^2}$ or $\dfrac{1-3v^2}{v^3-v}\,dv = 4\dfrac{dx}{x}$

Hence $\displaystyle\int \frac{1-3v^2}{v^3-v}\,dv - 4\int\frac{dx}{x} = c$

or $\displaystyle -\int \frac{3v^2-1}{v^3-v}\,dv - 4\log x = c$

ie., $-\log(v^3-v) - \log x^4 = c$

ie., $\log\left[(v^3-v)x^4\right] = -c = \log k$ *(say)*

$\Rightarrow$ $(v^3-v)x^4 = k$, where $v = y/x$

$\therefore$ $\left(\dfrac{y^3}{x^3} - \dfrac{y}{x}\right)x^4 = k$ or $xy^3 - x^3y = k$

Thus $xy^3 - x^3y = k$, is the required solution.

>> We have $\dfrac{dy}{dx} = \dfrac{y}{x-\sqrt{xy}}$ ... (1)

(*Observe that both the numerator and denominator are homogeneous functions of degree* 1)

Put $y = vx \quad \therefore \quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Now (1) becomes, $v + x\dfrac{dv}{dx} = \dfrac{vx}{x-\sqrt{x\cdot vx}}$

ie., $v + x\dfrac{dv}{dx} = \dfrac{vx}{x(1-\sqrt{v})}$ or $x\dfrac{dv}{dx} = \dfrac{v}{1-\sqrt{v}} - v$

ie., $x\dfrac{dv}{dx} = \dfrac{v-v+v\sqrt{v}}{1-\sqrt{v}}$ or $x\dfrac{dv}{dx} = \dfrac{v\sqrt{v}}{1-\sqrt{v}}$

$\therefore$ $\dfrac{1-\sqrt{v}}{v\sqrt{v}}\,dv = \dfrac{dx}{x}$

$\Rightarrow \quad \int v^{-3/2}\, dv - \int \frac{dv}{v} - \int \frac{dx}{x} = c$

ie., $\quad \frac{v^{-1/2}}{-1/2} - \log v - \log x = c$

ie., $\quad \frac{2}{\sqrt{v}} + \log(vx) = -c = k$ (*say*) where $v = y/x$

Thus $\quad \mathbf{2\sqrt{x/y} + \log y = k}$ is the required solution.

>> We have $(y - x) = (y + x)\frac{dy}{dx}$

or $\quad \frac{dy}{dx} = \frac{y - x}{y + x}$ ...(1)

Put $\quad y = vx \quad \therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

Now (1) becomes, $v + x\frac{dv}{dx} = \frac{vx - x}{vx + x}$

ie., $\quad v + x\frac{dv}{dx} = \frac{x(v-1)}{x(v+1)}$ or $x\frac{dv}{dx} = \frac{v-1}{v+1} - v$

ie., $\quad x\frac{dv}{dx} = \frac{v - 1 - v^2 - v}{v+1}$ or $x\frac{dv}{dx} = -\frac{(1+v^2)}{(v+1)}$

$\therefore \quad \frac{v+1}{1+v^2}\, dv = -\frac{dx}{x}$

$\Rightarrow \quad \int \frac{v\, dv}{1+v^2} + \int \frac{dv}{1+v^2} + \int \frac{dx}{x} = c$

ie., $\quad \frac{1}{2}\log(1+v^2) + \tan^{-1} v + \log x = c$

ie., $\quad \log\left[\sqrt{1+v^2}\cdot x\right] + \tan^{-1} v = c$, where $v = y/x$

$\therefore \quad \log\left[\sqrt{1+(y^2/x^2)}\cdot x\right] + \tan^{-1}(y/x) = c$.

Thus $\quad \mathbf{\log\sqrt{x^2+y^2} + \tan^{-1}(y/x) = c}$, is the required solution.

6. Solve $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$

\>> We have $x\,dy = \left[y + \sqrt{x^2+y^2} \right] dx$

or $$\frac{dy}{dx} = \frac{y + \sqrt{x^2+y^2}}{x} \qquad \ldots (1)$$

Put $y = vx \quad \therefore \quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Now (1) becomes $v + x\dfrac{dv}{dx} = \dfrac{v\,x + \sqrt{x^2 + v^2 x^2}}{x}$

ie., $$v + x\frac{dv}{dx} = \frac{x\left[v + \sqrt{1+v^2} \right]}{x} \quad \text{or} \quad x\frac{dv}{dx} = \sqrt{1+v^2}$$

$$\therefore \quad \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{dv}{\sqrt{1+v^2}} - \int \frac{dx}{x} = c$$

ie., $\sinh^{-1} v - \log x = c$, where $v = y/x$.

Thus $\sinh^{-1}(y/x) - \log x = c$, is the required solution

Note : $\displaystyle\int \frac{dv}{\sqrt{1+v^2}}$ *is also equal to* $\log\left(v + \sqrt{1+v^2} \right)$

Then the solution becomes $\log\left[\dfrac{y}{x} + \sqrt{1 + (y^2/x^2)} \right] - \log x = c$

ie., $$\log\left[\frac{y + \sqrt{x^2+y^2}}{x} \Big/ x \right] = c = \log k \ \textit{(say)}$$

$$\Rightarrow \quad \frac{y + \sqrt{x^2+y^2}}{x^2} = k$$

Thus $y + \sqrt{x^2+y^2} = kx^2$, is the solution.

We have $x\dfrac{dy}{dx} = y - \dfrac{y^2}{x}$

or $$\frac{dy}{dx} = \frac{xy - y^2}{x^2} \qquad \ldots (1)$$

Put $y = vx \quad \therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

Now (1) becomes $v + x\frac{dv}{dx} = \frac{x \cdot vx - v^2x^2}{x^2}$

ie., $v + x\,\frac{dv}{dx} = \frac{x^2(v - v^2)}{x^2}$ or $x\,\frac{dv}{dx} = -v^2$

$\therefore \quad \frac{dv}{v^2} = -\frac{dx}{x}$ by separating the variables.

Hence $\int \frac{dv}{v^2} + \int \frac{dx}{x} = c$ *ie.,* $-\frac{1}{v} + \log x = c,$ where $v = \frac{y}{x}$

Thus $\mathbf{\log x - \frac{x}{y} = c}$, is the required solution.

Remark : *The presence of terms involving y/x or x/y in the equation, can at once be recognized as a possible homogeneous equation.*

We have $\frac{dy}{dx} = \frac{y}{x} + \sin\left(\frac{y}{x}\right)$ . . . (1)

Put $\frac{y}{x} = v$ or $y = vx \quad \therefore \quad \frac{dy}{dx} = v + x\,\frac{dv}{dx}$

Now (1) becomes $v + x\,\frac{dv}{dx} = v + \sin v$

ie., $x\,\frac{dv}{dx} = \sin v$ or $\frac{dv}{\sin v} = \frac{dx}{x}$

Hence $\int \operatorname{cosec} v\, dv - \int \frac{dx}{x} = c$

ie., $\log(\operatorname{cosec} v - \cot v) - \log x = c$

or $\log\left[\frac{\operatorname{cosec} v - \cot v}{x}\right] = c = \log k$ (*say*)

$\Rightarrow \quad \operatorname{cosec} v - \cot v = kx,$ where $v = y/x$

Thus $\mathbf{\operatorname{cosec}(y/x) - \cot(y/x) = kx}$, is the required solution.

>> We have $\dfrac{dy}{dx} = 1 + \dfrac{y}{x} + \dfrac{y^2}{x^2}$ . . . (1)

Put $\dfrac{y}{x} = v$ or $y = vx$ $\therefore$ $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Now (1) becomes $v + x\dfrac{dv}{dx} = 1 + v + v^2$

ie., $x\dfrac{dv}{dx} = 1 + v^2$ or $\dfrac{dv}{1+v^2} = \dfrac{dx}{x}$

Hence $\displaystyle\int \frac{dv}{1+v^2} - \int \frac{dx}{x} = c$

ie., $\tan^{-1} v - \log x = c$, where $v = y/x$

Thus $\mathbf{\tan^{-1}(y/x) - \log x = c}$, is the required solution.

>> The given equation can be written as,

$$\frac{dy}{dx} = \frac{y\sec^2(y/x) - x\tan(y/x)}{x\sec^2(y/x)} \qquad \ldots (1)$$

Put $\dfrac{y}{x} = v$ or $y = vx$ $\therefore$ $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Now (1) becomes, $v + x\dfrac{dv}{dx} = \dfrac{v\,x\sec^2 v - x\tan v}{x\sec^2 v}$

ie., $v + x\dfrac{dv}{dx} = \dfrac{x(v\sec^2 v - \tan v)}{x\sec^2 v}$

ie., $x\dfrac{dv}{dx} = \dfrac{v\sec^2 v - \tan v}{\sec^2 v} - v$

ie., $x\dfrac{dv}{dx} = \dfrac{v\sec^2 v - \tan v - v\sec^2 v}{\sec^2 v}$

ie., $x\dfrac{dv}{dx} = \dfrac{-\tan v}{\sec^2 v}$ or $\dfrac{\sec^2 v}{\tan v}dv = -\dfrac{dx}{x}$

Hence $\displaystyle\int \frac{\sec^2 v}{\tan v}\,dv + \int \frac{dx}{x} = c$

ie., $\log(\tan v) + \log x = c$

or $\log(\tan v \cdot x) = c = \log k$ *(say),*

$\Rightarrow$ $x \tan v = k$ where $v = y/x$

Thus $\boldsymbol{x \tan(y/x) = k}$, is the required solution.

[illegible]

>> The given equation can be written in the form

$$\frac{dy}{dx} = \frac{y}{x}\left[\log(y/x) + 1\right] \qquad \ldots(1)$$

Put $y = vx \;\therefore\; \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Now (1) becomes $v + x\dfrac{dv}{dx} = v(\log v + 1)$

ie., $x\dfrac{dv}{dx} = v \log v$ or $\dfrac{dv}{v \log v} = \dfrac{dx}{x}$

Hence $\displaystyle\int \frac{1/v}{\log v}\,dv - \int \frac{dx}{x} = c$

ie., $\log(\log v) - \log x = c = \log k$ *(say)*

ie., $\log(\log v) = \log k + \log x$

ie., $\log(\log v) = \log(kx)$

$\Rightarrow$ $\log v = kx$ where $v = y/x$

Thus $\boldsymbol{\log(y/x) = kx}$, is the required solution.

12. [illegible]

>> We have, $[x - y\log(y/x)]\,dx + x\log(y/x)\,dy = 0$

Hence $$\frac{dy}{dx} = \frac{y\log(y/x) - x}{x\log(y/x)} \qquad \ldots(1)$$

This is a homogeneous equation.

Put $y = vx \;\therefore\; \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Hence (1) becomes,

$$v + x\frac{dv}{dx} = \frac{v\,x\log v - x}{x\log v}$$

ie., $x\dfrac{dv}{dx} = \dfrac{v\log v - 1}{\log v} - v$

ie., $x\dfrac{dv}{dx} = \dfrac{v\log v - 1 - v\log v}{\log v}$

ie., $x\dfrac{dv}{dx} = \dfrac{-1}{\log v}$

or $\log v\,d\,v = -\dfrac{dx}{x}$

$\Rightarrow$ $\int \log v\cdot 1\;d\,v + \int\dfrac{d\,x}{x} = c$

ie., $\log v\cdot v - \int v\cdot\dfrac{1}{v}\,d\,v + \log x = c$

ie., $v\log v - v + \log x = c$

ie., $v(\log v - 1) + \log x = c$ where $v = y/x$

Thus $(y/x)\,[\log(y/x) - 1] + \log x = c$, is the required solution.

>> We have $x^2\,dy = -(x^2 + 7xy + 16y^2)\,dx$

or $\dfrac{dy}{dx} = -\dfrac{(x^2 + 7xy + 16y^2)}{x^2}$ . . . (1)

Put $y = vx \;\therefore\; \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Now (1) becomes $v + x\dfrac{dv}{dx} = -\dfrac{(x^2 + 7v\,x^2 + 16v^2\,x^2)}{x^2}$

ie., $v + x\dfrac{dv}{dx} = -(1 + 7v + 16v^2)$

ie., $x\dfrac{dv}{dx} = -(16v^2 + 8v + 1)$

or $\dfrac{dv}{16v^2 + 8v + 1} = -\dfrac{dx}{x}$ by separating the variables.

$\Rightarrow$ $\int\dfrac{dv}{(4v + 1)^2} + \int\dfrac{dx}{x} = c$

ie., $\dfrac{(4v+1)^{-1}}{-1\cdot 4}+\log x = c$

ie., $-\dfrac{1}{4}\left(\dfrac{1}{4v+1}\right)+\log x = c,$ where $v = y/x$

$\therefore$ $-\dfrac{1}{4}\left(\dfrac{x}{4y+x}\right)+\log x = c$ . . . (2)

This is the general solution of the equation.

We have the condition $y(1) = 1$ That is $y = 1$ when $x = 1$.

Hence (2) becomes $-\dfrac{1}{20}+0 = c \quad \therefore \quad c = -\dfrac{1}{20}$

Now (2) becomes $-\dfrac{1}{4}\left(\dfrac{x}{4y+x}\right)+\log x = -\dfrac{1}{20}$

ie., $5x-20\log x(4y+x) = 4y+x$

ie., $4x-4y-20\log x(4y+x) = 0$

Thus $\boldsymbol{x-y-5\log x(x+4y) = 0}$ is the required particular solution.

>> Put $y = vx \quad \therefore \quad \dfrac{dy}{dx} = v+x\dfrac{dv}{dx}$

Now the given equation becomes

$$v+x\frac{dv}{dx} = \frac{x+2vx}{2x+vx}$$

$$v+x\frac{dv}{dx} = \frac{1+2v}{2+v}$$

or $x\dfrac{dv}{dx} = \dfrac{1+2v}{2+v}-v$

ie., $x\dfrac{dv}{dx} = \dfrac{1-v^2}{2+v}$ or $\dfrac{2+v}{1-v^2}\,dv = \dfrac{dx}{x}$

$\therefore$ $\displaystyle\int\frac{2+v}{1-v^2}\,dv-\int\frac{dx}{x} = c$

$$2\int\frac{dv}{1-v^2}+\int\frac{dv}{1-v^2}-\log x = c$$

$$2\cdot\frac{1}{2}\log\left(\frac{1+v}{1-v}\right)-\frac{1}{2}\log(1-v^2)-\log x = c$$

$$\log\left(\frac{1+v}{1-v}\right)-\log[\sqrt{1-v^2}\cdot x] = c$$

$$\log\left[\frac{1+v}{(1-v)\sqrt{1-v^2}\,x}\right] = \log k \ (say)$$

$\Rightarrow$ $\quad 1+v = kx(1-v)\sqrt{1-v^2}$

ie., $\quad 1+v = kx(1-v)\sqrt{1-v}\ \sqrt{1+v}$

ie., $\quad \sqrt{1+v} = kx(1-v)\sqrt{1-v}$ where $v = y/x$

ie., $\quad \sqrt{(x+y)/x} = kx\left(\frac{x-y}{x}\right)\sqrt{(x-y)/x}$

ie., $\quad \sqrt{x+y} = k(x-y)^{3/2}$

Thus $\quad x+y = k^2(x-y)^3$, is the required solution.

(As we observe terms with x/y , we need to express the equation relating to dx/dy and the terms are homogeneous functions of degree 0)

>> We have $(1+e^{x/y})\,dx = e^{x/y}\left(\frac{x}{y}-1\right)dy$

or $$\frac{dx}{dy} = \frac{e^{x/y}\left(\frac{x}{y}-1\right)}{(1+e^{x/y})} \qquad \ldots(1)$$

Put $\frac{x}{y} = v$ or $x = vy$ $\therefore$ $\frac{dx}{dy} = v+y\frac{dv}{dy}$

Now (1) becomes, $v+y\frac{dv}{dy} = \frac{e^v(v-1)}{(1+e^v)}$

ie., $\quad y\frac{dv}{dy} = \frac{e^v(v-1)}{1+e^v}-v$ or $y\frac{dv}{dy} = \frac{e^v v-e^v-v-e^v v}{1+e^v}$

ie., $\quad y\frac{dv}{dy} = -\frac{(e^v+v)}{(1+e^v)}$ or $\frac{(1+e^v)\,dv}{e^v+v} = -\frac{dy}{y}$

Hence $\int\frac{(1+e^v)\,dv}{e^v+v}+\int\frac{dy}{y} = c$

ie., $\quad \log(e^v+v)+\log y = c$

or $\quad \log\left[\,(e^v + v)\,y\,\right] = \log k \;\; (say)$

$\Rightarrow \quad (e^v + v)\,y = k, \;\;$ where $v = x/y$

Thus $\quad y\,e^{x/y} + x = k,$ is the required solution.

[illegible]

>> We have $[\,x + y\cot(x/y)\,]\,dy = y\,dx$

$$\therefore \quad \frac{dx}{dy} = \frac{x + y\cot(x/y)}{y} \qquad \ldots (1)$$

Put $\quad \dfrac{x}{y} = v$ or $x = vy \;\; \therefore \;\; \dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$

Now (1) becomes, $v + y\dfrac{dv}{dy} = \dfrac{vy + y\cot v}{y}$

ie., $\quad v + y\dfrac{dv}{dy} = \dfrac{y(v + \cot v)}{y}$ or $y\dfrac{dv}{dy} = \cot v$

$\therefore \quad \dfrac{dv}{\cot v} = \dfrac{dy}{y}$ by separating the variables.

$\Rightarrow \quad \displaystyle\int \tan v\, dv - \int \frac{dy}{y} = c$

ie., $\quad \log(\sec v) - \log y = c$ or $\log\left(\dfrac{\sec v}{y}\right) = \log k \;\; (say)$

Hence we have $\sec v = ky,$ where $v = x/y$

Thus $\quad \sec(x/y) = ky,$ is the required solution.

17. [illegible]

>> We have $[\,y + x\sin^2(x/y)\,]\,dy = y\sin^2(x/y)\,dx$

$$\therefore \quad \frac{dx}{dy} = \frac{y + x\sin^2(x/y)}{y\sin^2(x/y)} \qquad \ldots (1)$$

Put $\quad \dfrac{x}{y} = v$ or $x = vy \;\; \therefore \;\; \dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$

Now (1) becomes, $v + y\dfrac{dv}{dy} = \dfrac{y + vy\sin^2 v}{y\sin^2 v}$

ie., $\quad v + y\dfrac{dv}{dy} = \dfrac{y(1 + v\sin^2 v)}{y\sin^2 v}$ or $y\dfrac{dv}{dy} = \dfrac{1 + v\sin^2 v}{\sin^2 v} - v$

ie., $\quad y\dfrac{dv}{dy} = \dfrac{1 + v\sin^2 v - v\sin^2 v}{\sin^2 v} \quad$ or $\quad y\dfrac{dv}{dy} = \dfrac{1}{\sin^2 v}$

$\therefore \quad \sin^2 v\,dv = \dfrac{dy}{y}$ by separating the variables.

$$\int \sin^2 v\,dv - \int \frac{dy}{y} = c \quad \text{or} \quad \int \frac{1-\cos 2v}{2}\,dv - \log y = c$$

ie., $\quad \dfrac{1}{2}\displaystyle\int dv - \dfrac{1}{2}\int \cos 2v\,dv - \log y = c$

ie., $\quad \dfrac{v}{2} - \dfrac{\sin 2v}{4} - \log y = c$, where $v = x/y$

Thus $\quad \dfrac{x}{2y} - \dfrac{1}{4}\sin(2x/y) - \log y = c \quad$ is the required solution.

Consider the differential equation in the form :

$$(ax + by + c)\,dx \pm (a'x + b'y + c')\,dy = 0$$

We first express the equation in respect of $\dfrac{dy}{dx}$ and the procedure is narrated by taking

$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'} \quad \text{where } \frac{a}{a'} \neq \frac{b}{b'} \qquad \ldots (1)$$

This condition implies that there are no common factors for the x and y terms in the numerator as well as in the denominator.

Put $x = X + h,\ y = Y + k$ where h and k are constants to be chosen appropriately later.

Now $\quad \dfrac{dy}{dx} = \dfrac{dy}{dY}\,\dfrac{dY}{dX}\,\dfrac{dX}{dx}$

$$= 1 \cdot \frac{dY}{dX} \cdot 1 \quad \text{Hence} \quad \frac{dy}{dx} = \frac{dY}{dX}$$

As a consequence of these (1) becomes

$$\frac{dY}{dX} = \frac{a(X+h) + b(Y+k) + c}{a'(X+h) + b'(Y+k) + c'}$$

ie., $$\frac{dY}{dX} = \frac{(aX + bY) + (ah + bk + c)}{(a'X + b'Y) + (a'h + b'k + c')} \qquad \ldots (2)$$

Now, let us choose h and k such that :

$$ah + bk + c = 0 \quad \text{and} \quad a'h + b'k + c' = 0$$

Solving these equations we get the value for h and k.

Thus (2) now assumes the form

$$\frac{dY}{dX} = \frac{aX + bY}{a'X + b'Y} \qquad \ldots (3)$$

It is evident that (3) is a homogeneous equation in the variables X and Y. This equation can be solved by putting $Y = VX$ as discussed already. Finally we substitute for X and Y where $X = x - h$, $Y = y - k$.

Remark : If $\frac{a}{a'} = \frac{b}{b'} = k$ (say) *then* $a = a'k$, $b = b'k$.

Hence (1) *becomes* $\frac{dy}{dx} = \frac{a'kx + b'ky + c}{a'x + b'y + c'}$

or $$\frac{dy}{dx} = \frac{k(a'x + b'y) + c}{(a'x + b'y) + c'}$$

We can solve by putting $t = a'x + b'y$. *(Reducible to variables separable)*

>> We have $(4x + y - 2)\,dy = -(x - 4y - 9)\,dx$

$$\therefore \quad \frac{dy}{dx} = \frac{-x + 4y + 9}{4x + y - 2} \qquad \ldots (1)$$

Put $x = X + h$ and $y = Y + k$ where h and k are constants to be chosen suitably later.

Now $\frac{dy}{dx} = \frac{dy}{dY}\frac{dY}{dX}\frac{dX}{dx}$

$= 1 \cdot \frac{dY}{dX} \cdot 1 \qquad$ Hence $\frac{dy}{dx} = \frac{dY}{dX}$

Thus (1) becomes,

$$\frac{dY}{dX} = \frac{-(X + h) + 4(Y + k) + 9}{4(X + h) + (Y + k) - 2}$$

ie., $$\frac{dY}{dX} = \frac{(-X + 4Y) + (-h + 4k + 9)}{(4X + Y) + (4h + k - 2)} \qquad \ldots (2)$$

Let us choose h and k such that

$$-h+4k+9 = 0 \text{ and } 4h+k-2 = 0$$

Solving these equations we get, $h = 1$ and $k = -2$

Thus (2) becomes $\dfrac{dY}{dX} = \dfrac{-X+4Y}{4X+Y}$ ... (3)

Put $Y = VX \Rightarrow \dfrac{dY}{dX} = V + X\dfrac{dV}{dx}$

Now (3) becomes, $V + X\dfrac{dV}{dX} = \dfrac{-X+4VX}{4X+VX}$

ie., $V + X\dfrac{dV}{dX} = \dfrac{X(-1+4V)}{X(4+V)}$ or $X\dfrac{dV}{dX} = \dfrac{-1+4V}{4+V} - V$

ie., $X\dfrac{dV}{dX} = \dfrac{-1+4V-4V-V^2}{4+V}$ ie., $X\dfrac{dV}{dX} = \dfrac{-(1+V^2)}{(4+V)}$

$\therefore$ $\dfrac{(4+V)\,dV}{1+V^2} = -\dfrac{dX}{X}$ by separating the variables.

$\Rightarrow$ $4\displaystyle\int\frac{dV}{1+V^2} + \int\frac{V\,dV}{1+V^2} + \int\frac{dX}{X} = c$

ie., $4\tan^{-1}V + \dfrac{1}{2}\log(1+V^2) + \log X = c$

ie., $8\tan^{-1}V + \log(1+V^2) + 2\log X = 2c$

ie., $8\tan^{-1}V + \log\left[(1+V^2)X^2\right] = 2c$, where $V = Y/X$.

$\therefore$ $8\tan^{-1}(Y/X) + \log(X^2+Y^2) = 2c = k$ (say)

But $X = x-h = x-1$ and $Y = y-k = y+2$

Thus $8\tan^{-1}\left(\dfrac{y+2}{x-1}\right) + \log\left[(x-1)^2+(y+2)^2\right] = k$, is the required solution.

>> We have $\dfrac{dy}{dx} = \dfrac{x+y-1}{x-y+1}$ ... (1)

Put $x = X+h$ and $y = Y+k$, where h and k are constants to be chosen suitably later.

$\therefore \quad \dfrac{dy}{dx} = \dfrac{dy}{dY}\,\dfrac{dY}{dX}\,\dfrac{dX}{dx}$

$= 1 \cdot \dfrac{dY}{dX} \cdot 1 \quad$ Hence $\dfrac{dy}{dx} = \dfrac{dY}{dX}$

Thus (1) becomes,

$$\frac{dY}{dX} = \frac{(X+h)+(Y+k)-1}{(X+h)-(Y+k)+1}$$

ie., $$\frac{dY}{dX} = \frac{(X+Y)+(h+k-1)}{(X-Y)+(h-k+1)} \qquad \ldots (2)$$

Let us choose h and k such that

$$h+k-1 = 0 \text{ and } h-k+1 = 0$$

Solving these equations we get, $h = 0,\ k = 1.$

Thus (2) becomes, $\dfrac{dY}{dX} = \dfrac{X+Y}{X-Y} \qquad \ldots (3)$

Put $\quad Y = VX \ \therefore\ \dfrac{dY}{dX} = V + X\,\dfrac{dV}{dX}$

Now (3) becomes, $V + X\,\dfrac{dV}{dX} = \dfrac{X+VX}{X-VX}$

ie., $\quad V + X\,\dfrac{dV}{dX} = \dfrac{X(1+V)}{X(1-V)} \quad$ or $\quad X\,\dfrac{dV}{dX} = \dfrac{1+V}{1-V} - V$

ie., $\quad X\,\dfrac{dV}{dX} = \dfrac{1+V-V+V^2}{1-V} \quad$ *ie.,* $\quad X\,\dfrac{dV}{dX} = \dfrac{1+V^2}{1-V}$

$\therefore \quad \dfrac{1-V}{1+V^2}\,dV = \dfrac{dX}{X} \quad$ by separating the variables.

$\Rightarrow \quad \displaystyle\int \frac{dV}{1+V^2} - \int \frac{V\,dV}{1+V^2} - \int \frac{dX}{X} = c$

ie., $\quad \tan^{-1} V - \dfrac{1}{2}\log(1+V^2) - \log X = c$

ie., $\quad 2\tan^{-1} V - \log(1+V^2) - 2\log X = 2c$

ie., $\quad 2\tan^{-1} V - \log\left[(1+V^2)X^2\right] = 2c$, where $V = Y/X$

ie., $\quad 2\tan^{-1}(Y/X) - \log\left[X^2+Y^2\right] = 2c = k$ (*say*)

But $\quad X = x - h = x$ and $Y = y - k = y - 1$

Thus $\quad \mathbf{2\tan^{-1}\left(\dfrac{y-1}{x}\right) - \log\left[x^2+(y-1)^2\right] = k}$ is the required solution.

20. Solve: $(3y - 7x + 7)\,dx + (7y - 3x + 3)\,dy = 0$

>> We have $(7y - 3x + 3)\,dy = (7x - 3y - 7)\,dx$

$$\therefore \quad \frac{dy}{dx} = \frac{7x - 3y - 7}{-3x + 7y + 3} \qquad \ldots (1)$$

Put $x = X + h$ and $y = Y + k$, where h and k are constants to be chosen suitably later.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dY}\,\frac{dY}{dX}\,\frac{dX}{dx}$$

$$= 1 \cdot \frac{dY}{dX} \cdot 1 \quad \text{Hence } \frac{dy}{dx} = \frac{dY}{dX}$$

Thus (1) becomes,

$$\frac{dY}{dX} = \frac{7(X+h) - 3(Y+k) - 7}{-3(X+h) + 7(Y+k) + 3}$$

ie., $$\frac{dY}{dX} = \frac{(7X - 3Y) + (7h - 3k - 7)}{(-3X + 7Y) + (-3h + 7k + 3)} \qquad \ldots (2)$$

Let us choose h and k such that,

$$7h - 3k - 7 = 0 \text{ and } -3h + 7k + 3 = 0$$

Solving these equations we get, $h = 1,\ k = 0.$

Thus (2) becomes, $\dfrac{dY}{dX} = \dfrac{7X - 3Y}{-3X + 7Y}$ ... (3)

Put $Y = VX,\ \therefore\ \dfrac{dY}{dX} = V + X\dfrac{dV}{dX}$

Now (3) becomes $V + X\dfrac{dV}{dX} = \dfrac{7X - 3VX}{-3X + 7VX}$

ie., $$V + X\frac{dV}{dX} = \frac{X(7 - 3V)}{X(-3 + 7V)}$$

ie., $$X\frac{dV}{dX} = \frac{7 - 3V}{-3 + 7V} - V \text{ or } X\frac{dV}{dX} = \frac{7 - 7V^2}{-3 + 7V}$$

$$\therefore \quad \frac{-3 + 7V}{1 - V^2}\,dV = 7\,\frac{dX}{X} \text{ by separating the variables.}$$

$$\Rightarrow \quad -3\int \frac{dV}{1 - V^2} + 7\int \frac{V\,dV}{1 - V^2} - 7\int \frac{dX}{X} = c$$

Using $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log\left(\frac{a+x}{a-x}\right)$ for the first term we have,

$$-\frac{3}{2}\log\left(\frac{1+V}{1-V}\right) - \frac{7}{2}\log(1 - V^2) - 7\log X = c$$

or $$3\log\left(\frac{1+V}{1-V}\right) + 7\log(1 - V^2) + 14\log X = -2c$$

or $$\log\left[\left(\frac{1+V}{1-V}\right)^3 (1 - V^2)^7 X^{14}\right] = \log c_1 \ (say)$$

$\Rightarrow$ $$\frac{(1+V)^3}{(1-V)^3}(1-V)^7(1+V)^7 X^{14} = c_1$$

or $$(1+V)^{10}(1-V)^4 X^{14} = c_1$$

Taking square root we have,

$$(1+V)^5(1-V)^2 X^7 = \sqrt{c_1}, \quad \text{where } V = Y/X$$

ie., $$\left(\frac{X+Y}{X}\right)^5 \left(\frac{X-Y}{X}\right)^2 X^7 = k \ (say) \text{ where } k = \sqrt{c_1}$$

ie., $$(X+Y)^5(X-Y)^2 = k$$

But $X = x - h = x - 1$ and $Y = y - k = y$

Thus $(x+y-1)^5(x-y-1)^2 = k$, is the required solution.

21. Solve : $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$

>> We have $$\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3} \quad \ldots (1)$$

Put $x = X + h$ and $y = Y + k$, where h and k are constants to be chosen suitably later.

$\therefore$ $$\frac{dy}{dx} = \frac{dy}{dY}\frac{dY}{dX}\frac{dX}{dx}$$

$$= 1 \cdot \frac{dY}{dX} \cdot 1 \quad \text{Hence } \frac{dy}{dx} = \frac{dY}{dX}$$

Thus (1) becomes, $$\frac{dY}{dX} = \frac{(X+h) + 2(Y+k) - 3}{2(X+h) + (Y+k) - 3}$$

ie., $$\frac{dY}{dX} = \frac{(X+2Y)+(h+2k-3)}{(2X+Y)+(2h+k-3)} \quad \ldots (2)$$

Let us choose h and k such that,

$$h+2k-3 = 0 \text{ and } 2h+k-3 = 0$$

Solving these equations we get $h = 1$ and $k = 1$

Thus (2) becomes $\dfrac{dY}{dX} = \dfrac{X+2Y}{2X+Y}$ ... (3)

Put $Y = VX \quad \therefore \quad \dfrac{dY}{dX} = V + X\dfrac{dV}{dX}$

Now (3) becomes $V + X\dfrac{dV}{dX} = \dfrac{X+2VX}{2X+VX}$

ie., $$V + X\frac{dV}{dX} = \frac{X(1+2V)}{X(2+V)}$$

or $$X\frac{dV}{dX} = \frac{1+2V}{2+V} - V \text{ or } X\frac{dV}{dX} = \frac{1-V^2}{2+V}$$

$\therefore$ $\dfrac{2+V}{1-V^2}dV = \dfrac{dX}{X}$ by separating the variables.

$$\Rightarrow \quad 2\int\frac{dV}{1-V^2} + \int\frac{V\,dV}{1-V^2} - \int\frac{dX}{X} = c$$

ie., $$2\cdot\frac{1}{2}\log\left(\frac{1+V}{1-V}\right) - \frac{1}{2}\log(1-V^2) - \log X = c$$

ie., $$2\log\left(\frac{1+V}{1-V}\right) - \log(1-V^2) - 2\log X = 2c$$

ie., $$\log\left(\frac{1+V}{1-V}\right)^2 - \log\left[(1-V^2)X^2\right] = 2c$$

or $$\log\left[\frac{(1+V)^2}{(1-V)^2(1-V^2)X^2}\right] = \log k \text{ (say)}$$

$$\Rightarrow \quad \frac{(1+V)}{(1-V)^3X^2} = k \quad \text{or} \quad \frac{(X+Y)}{(X-Y)^3} = k \text{ since } V = \frac{Y}{X}$$

But $X = x-h = x-1$ and $Y = y-k = y-1$

Thus $(x+y-2) = k(x-y)^3$, **is the required solution.**

22. Solve : $\frac{dy}{dx} = \frac{y-x+5}{y+x+3}$

>> We have $\frac{dy}{dx} = \frac{y-x+5}{y+x+3}$ ... (1)

Put $x = X+h$ and $y = Y+k$, where h and k are constants to be chosen later.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dY}\,\frac{dY}{dX}\,\frac{dX}{dx}$$

$$= 1 \cdot \frac{dY}{dX} \cdot 1 \quad \text{Hence } \frac{dy}{dx} = \frac{dY}{dX}$$

Thus (1) becomes

$$\frac{dY}{dX} = \frac{(Y+k)-(X+h)+5}{(Y+k)+(X+h)+3}$$

$$\frac{dY}{dX} = \frac{(Y-X)+(k-h+5)}{(Y+X)+(k+h+3)} \qquad \ldots (2)$$

Let us choose h and k such that

$$k-h+5 = 0 \text{ and } k+h+3 = 0$$

By solving these equations we get $h = 1$ and $k = -4$.

Thus (2) becomes $\frac{dY}{dX} = \frac{Y-X}{Y+X}$ ... (3)

Put $Y = VX \quad \therefore \quad \frac{dY}{dX} = V + X\frac{dV}{dX}$

Now (3) becomes,

$$V + X\frac{dV}{dX} = \frac{VX-X}{VX+X}$$

or $X\frac{dV}{dX} = \frac{V-1}{V+1} - V$ or $X\frac{dV}{dX} = -\frac{(1+V^2)}{V+1}$

$\therefore \quad \frac{V+1}{1+V^2}\,dV = \frac{-dX}{X}$ by separating the variables.

$$\Rightarrow \quad \int \frac{V+1}{1+V^2}\,dV + \int \frac{dX}{X} = c$$

ie., $\int \frac{VdV}{1+V^2} + \int \frac{dV}{1+V^2} + \log X = c$

ie., $\frac{1}{2}\log(1+V^2) + \tan^{-1} V + \log X = c$

ie., $\log [X\sqrt{1+V^2}] + \tan^{-1} V = c$ where $V = Y/X$

ie., $\log [X\sqrt{1+(Y^2/X^2)}] + \tan^{-1}(Y/X) = c$

ie., $\log \sqrt{X^2+Y^2} + \tan^{-1}(Y/X) = c$

But $X = x - h = x - 1$ and $Y = y - k = y + 4$

Thus $\log \sqrt{(x-1)^2+(y+4)^2} + \tan^{-1}[(y+4)/(x-1)] = c$,
is the required solution.

[illegible] particular solution of the equation
[illegible] $(3x - 2y - 5)$ [illegible] $= 0$ subject to the condition $y(1) = 2$

>> We have $(x-1)\,dx = (3x-2y-5)\,dy$

$$\therefore \quad \frac{dy}{dx} = \frac{x-1}{3x-2y-5} \qquad \ldots (1)$$

Put $x = X + h$ and $y = Y + k$ where h and k are constants to be chosen later.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dY} \cdot \frac{dY}{dX} \cdot \frac{dX}{dx}$$

$$= 1 \cdot \frac{dY}{dX} \cdot 1 \qquad \text{Hence } \frac{dy}{dx} = \frac{dY}{dX}$$

Thus (1) becomes $\dfrac{dY}{dX} = \dfrac{(X+h)-1}{3(X+h)-2(Y+k)-5}$

$$\text{ie.,} \quad \frac{dY}{dX} = \frac{X+(h-1)}{(3X-2Y)+(3h-2k-5)} \qquad \ldots (2)$$

Let us choose h and k such that

$$h - 1 = 0 \text{ and } 3h - 2k - 5 = 0$$

These equations will give us $h = 1,\ k = -1$ on solving.

Thus (2) becomes, $\dfrac{dY}{dX} = \dfrac{X}{3X-2Y}$ $\qquad \ldots (3)$

Put $Y = VX \quad \therefore \quad \dfrac{dY}{dX} = V + X\dfrac{dV}{dX}$

Now (3) becomes, $V + X\dfrac{dV}{dX} = \dfrac{X}{3X-2VX}$

ie, $V + X\dfrac{dV}{dX} = \dfrac{X}{X(3-2V)}$

or $X\dfrac{dV}{dX} = \dfrac{1}{3-2V} - V$

ie., $X\dfrac{dV}{dX} = \dfrac{1-3V+2V^2}{3-2V}$

$\therefore \quad \dfrac{3-2V}{1-3V+2V^2}\,dV = \dfrac{dX}{X}$ by separating the variables.

$$\Rightarrow \quad \int \frac{3-2V}{2V^2-3V+1} - \int \frac{dX}{X} = c \qquad \ldots (4)$$

We have $2V^2-3V+1 = (2V-1)(V-1)$ by factorization.

Now, let $\dfrac{3-2V}{(2V-1)(V-1)} = \dfrac{A}{2V-1} + \dfrac{B}{V-1}$

or $\quad 3-2V = A(V-1)+B(2V-1)$

Put $\quad V = 1 \quad : 1 = B(1) \qquad \therefore\ B = 1$

Put $\quad V = 1/2 \quad : 2 = A(-1/2) \qquad \therefore\ A = -4$

Now $\quad \displaystyle\int \frac{(3-2V)\,dV}{(2V-1)(V-1)} = -4\int \frac{dV}{2V-1} + \int \frac{dV}{V-1}$

$$\text{ie.,} \quad \int \frac{(3-2V)\,dV}{2V^2-3V+1} = -2\log(2V-1) + \log(V-1) \qquad \ldots (5)$$

Using (5) in (4) we have

$$-2\log(2V-1) + \log(V-1) - \log X = c$$

or $\quad \log\left[\dfrac{V-1}{(2V-1)^2 X}\right] = \log k$ *(say)*

$\Rightarrow \quad \dfrac{V-1}{(2V-1)^2 X} = k$ where $V = \dfrac{Y}{X}$

$\therefore \quad \dfrac{Y-X}{(2Y-X)^2} = k$ or $(Y-X) = k(2Y-X)^2$

But $\quad X = x-h = x-1$ and $Y = y-k = y+1$

$\therefore \quad (y-x+2) = k(2y-x+3)^2$ is the general solution of the equation.

Now consider $y(1) = 2$. That is $y = 2$ when $x = 1$.

The general solution becomes $3 = k(36)$ or $k = 1/12$

Thus $\quad \mathbf{12(y-x+2) = (2y-x+3)^2}$ is the required particular solution.

24. Solve $(2x + y - 1)\,dy = (x - 2y + 5)\,dx$

>> $$\frac{dy}{dx} = \frac{x - 2y + 5}{2x + y - 1} \qquad \ldots (1)$$

Put $x = X + h$ and $y = Y + k$, where h and k are constants to be chosen later.

$$\frac{dy}{dx} = \frac{dy}{dY}\,\frac{dY}{dX}\,\frac{dX}{dx}$$

$$\frac{dy}{dx} = 1 \cdot \frac{dY}{dX} \cdot 1 = \frac{dY}{dX}$$

Hence (1) becomes,

$$\frac{dY}{dX} = \frac{(X + h) - 2(Y + k) + 5}{2(X + h) + (Y + k) - 1}$$

ie., $$\frac{dY}{dX} = \frac{(X - 2Y) + (h - 2k + 5)}{(2X + Y) + (2h + k - 1)} \qquad \ldots (2)$$

Let us choose h and k such that,

$$h - 2k + 5 = 0 \text{ and } 2h + k - 1 = 0$$

Solving these equations we get

$$h = -3/5 \text{ and } k = 11/5$$

Now (2) becomes,

$$\frac{dY}{dX} = \frac{X - 2Y}{2X + Y} \qquad \ldots (3)$$

Put $Y = VX$

$$\therefore \quad \frac{dY}{dX} = V + X\,\frac{dV}{dX}$$

Hence (3) becomes,

$$V + X\frac{dV}{dX} = \frac{X(1 - 2V)}{X(2 + V)} \quad \text{or} \quad X\,\frac{dV}{dX} = \frac{1 - 2V}{2 + V} - V$$

ie., $$X\,\frac{dV}{dX} = \frac{1 - 4V - V^2}{2 + V} \quad \text{or} \quad \frac{2 + V}{1 - 4V - V^2}\,dV = \frac{dX}{X},$$ by separating the variables.

$$\Rightarrow \quad \int \frac{2 + V}{1 - 4V - V^2}\,dV - \frac{dX}{X} = c$$

ie., $$-\frac{1}{2}\log(1 - 4V - V^2) - \log X = c$$

or $\qquad \log[(1-4V-V^2)X^2] = -2c = \log c_1 \ (say)$

$\Rightarrow \qquad (1-4V-V^2)X^2 = c_1$, where $V = \dfrac{Y}{X}$

ie., $\qquad \left(1-\dfrac{4Y}{X}-\dfrac{Y^2}{X^2}\right)X^2 = c_1 \quad$ or $\quad X^2-4XY-Y^2 = c_1 \qquad \ldots(4)$

But $X = x-h = x+(3/5)$ and $Y = y-k = y-(11/5)$

Using these in (4) and simplifying we obtain the required solution in the form

$$\mathbf{x^2-4xy-y^2+10x+2y = k}$$

EXERCISES

Solve the following equations.

1. $(xy+y^2)\,dx - x^2\,dy = 0$
2. $(x^3+y^3)\,dx = xy(x+y)\,dy$
3. $y\,dx - x\,dy = \sqrt{y^2-x^2}\,dy$
4. $(2x+y)^2 = xy\dfrac{dy}{dx}$
5. $x\,dy = [y - x\cos^2(y/x)]\,dx$
6. $\dfrac{dy}{dx}+\dfrac{x^3+3xy^2}{y^3+3x^2y} = 0$
7. $x[dx+\sin(y/x)\,dy] = y\sin(y/x)\,dx$
8. $[x\,dy - y\,dx]\,y\sin(y/x) = [x\,dy+y\,dx]\,x\cos(y/x)$
9. $(x+y\log x - y\log y)\,dx - x(\log x-\log y)\,dy = 0$
10. $e^{x/y}[y\,dx - x\,dy] = y^2\,dy$
11. $(x^2-4xy-2y^2)\,dx - (2x^2+4xy-y^2)\,dy = 0\ ;\ y(0) = 2$
12. $(y-x-4)\,dx = (y+x-2)\,dy$
13. $(y-2)\,dx+(y-x+1)\,dy = 0\ ;\ y(5) = 3$
14. $(3x-y-1)\,dy+2(x-y)\,dx = 0$
15. $\dfrac{dy}{dx}+\dfrac{2x+3y}{y+2} = 0$

1 $x/y + \log x = c$

2. $y/x + \log(y - x) = cx$

3. $\sin^{-1}(x/y) - \log y = c$

4. $\log(1 + y/x) + 4(\log x - 1/x) = c$

5. $\log x + \tan(y/x) = c$

6. $x^4 + 6x^2y^2 + y^4 = c$

7. $\log x - \cos(y/x) = c$

8. $xy\cos(y/x) = c$

9. $\log x + (y/x)[\log(y/x) - 1] = c$

10. $e^{x/y} - y = c$

11. $x^3 - 6x^2y - 6xy^2 + y^3 = 8$

12. $\log[(x+1)^2 + (y-3)^2] + 2\tan^{-1}[(y-3)/(x+1)] = c$

13. $\log(y-2) + \{(x-3) \quad (y-2)\} = 2$

14. $(x + y - 1)^4 = c(4x - 2y - 1)$

15. $(2x + y - 4)^2 = c(x + y - 1)$

6.34 Exact Differential Equations

We introduce this topic with an illustration.

Consider a function $f(x, y) = x^2 + xy + y^2 + x + y = c$ where c is an arbitrary constant.

Let us take the differential of this function.

ie., $df = 2x\,dx + x\,dy + y\,dx + 2y\,dy + dx + dy = 0$

ie., $(2x + y + 1)\,dx + (x + 2y + 1)\,dy = 0$

Obviously we can say that the solution of this differential equation is $f(x, y) = c$ where $f(x, y)$ is the function we started with.

In other words, if $M(x, y) = 2x + y + 1$ and $N(x, y) = x + 2y + 1$ then

$$d[f(x, y)] = M(x, y)\,dx + N(x, y)\,dy$$

$\therefore$ the solution of the equation $M(x, y)\,dx + N(x, y)\,dy = 0$ is equivalent to the solution of the equation

$d[f(x, y)] = 0$ which is $f(x, y) = c$, on integration.

In other words, if we are able to identify that in a given differential equation :

$$M(x, y)\,dx + N(x, y)\,dy = 0,$$

$$M(x, y)\,dx + N(x, y)\,dy = d\,[f(x, y)]$$

then we can simply conclude that the solution of the equation is $f(x, y) = c$.

Thus we say that $M(x, y)\,dx + N(x, y)\,dy = 0$ *is an exact differential equation if there exists a function* $f(x, y)$ *such that*

$$df = M(x, y)\,dx + N(x, y)\,dy.$$

How to identify that the given equation is exact? If so what is its solution? These two questions are answered in the following **theorem**.

Statement : The necessary and the suffecient condition for the differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ to be an exact equation is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Further **the solution of the exact equation is** given by

$$\int M\,dx + \int N(y)\,dy = c$$

where, in the first term we integrate $M(x, y)$ $w.r.t$ x keeping y fixed and $N(y)$ indicate the terms in N without x (*not containing* x)

Important Note : *This method is a very easy method for solving a differential equation. Some of the equations of the types : reducible to the variables separable form, homogeneous and reducible to the homogeneous form might be an exact equation. Therefore it is very much advisible that if an equation without the involvement of* y/x *or* x/y *terms is homogeneous then exactness is checked before proceeding to solve by putting* $y = vx$ *or* $x = vy$. *However in the case of terms involving* y/x *or* x/y *homogeneous equation solving procedure itself is easier. This method should be tried in the case of problems which are of the type*

$$(ax + by + c)\,dx \pm (a'x + b'y + c')\,dy = 0.$$

First few problems in this method to follow belong to these category.

WORKED PROBLEMS

25. Solve : $(2x + y + 1)\,dx + (x + 2y + 1)\,dy = 0$

Note : *The given equation can be written in the form* $\dfrac{dy}{dx} = -\dfrac{2x + y + 1}{x + 2y + 1}$ *and we have already discussed the method of solving this equation by putting* $x = X + h$, *and* $y = Y + k$. *Before one ventures to do so, the condition for an exact* $d \cdot e$ *be checked by just scribbling.*

>> Let $M = 2x + y + 1$ and $N = x + 2y + 1$

$\therefore \quad \dfrac{\partial M}{\partial y} = 1$ and $\dfrac{\partial N}{\partial x} = 1$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$, where $N(y)$ denotes terms in N not containing x.

ie., $\int (2x + y + 1)\,dx + \int (2y + 1)\,dy = c$

ie., $x^2 + xy + x + y^2 + y = c$

Thus $\boldsymbol{x^2 + xy + y^2 + x + y = c}$, is the required solution.

Remark : *This example is the illustrative example which we took to introduce the concept of an exact $d \cdot e$ and the solution that we have obtained is same as $f(x, y)$ in the illustration. The method is so simple as one can imagine the amount of work involved had it been done by reducing it to a homogeneous equation.*

---

26. *Solve* : $(y^3 - 3x^2 y)\,dx - (x^3 - 3xy^2)\,dy = 0$

Note : *Observe that this is a homogeneous equation.*

\>> Let $M = y^3 - 3x^2 y$ and $N = -x^3 + 3xy^2$

$\therefore$ $\dfrac{\partial M}{\partial y} = 3y^2 - 3x^2$ and $\dfrac{\partial N}{\partial x} = -3x^2 + 3y^2$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $\int (y^3 - 3x^2 y)\,dx + \int 0\,dy = c$

(*Observe that there is no term in* N *which do not contain* x)

$\therefore$ $y^3 \cdot x - x^3 y = c$

Thus $\boldsymbol{xy^3 - x^3 y = c}$, is the required solution.

Remark : *Compare the working and the answer of this Problem with Problem - 3 solved by putting* $y = vx$.

---

27. *Solve* : $(5x^4 + 3x^2 y^2 - 2xy^3)\,dx + (2x^3 y - 3x^2 y^2 - 5y^4)\,dy = 0$

[*Though it is evident that the equation is a homogeneous one, before solving by putting* $y = vx$ *we should check for exactness*]

\>> Let $M = 5x^4 + 3x^2 y^2 - 2xy^3$ and $N = 2x^3 y - 3x^2 y^2 - 5y^4$

$$\frac{\partial M}{\partial y} = 6x^2 y - 6xy^2 \text{ and } \frac{\partial N}{\partial x} = 6x^2 y - 6xy^2$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact .

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $\int \left(5x^4 + 3x^2 y^2 - 2xy^3\right) dx + \int -5y^4\,dy = c$

Thus $x^5 + x^3 y^2 - x^2 y^3 - y^5 = c$, is the required solution.

Note : *The problem can be solved by writing the equation in the form*

$$\frac{dy}{dx} = -\frac{(x+3y)-4}{3(x+3y)-2} \text{ and then by putting } x + 3y = t.$$

>> The given equation is equivalent to the form

$$(x + 3y - 4)\,dx + (3x + 9y - 2)\,dy = 0$$

Let $M = x + 3y - 4$ and $N = 3x + 9y - 2$

$\therefore$ $\frac{\partial M}{\partial y} = 3$ and $\frac{\partial N}{\partial x} = 3$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $\int (x + 3y - 4)\,dx + \int (9y - 2)\,dy = c$

ie., $\frac{x^2}{2} + 3xy - 4x + \frac{9y^2}{2} - 2y = c$

Thus $x^2 + 6xy - 8x + 9y^2 - 4y = 2c$, is the required solution.

>> Let $M = y(1 + 1/x) + \cos y$ and $N = x + \log x - x \sin y$

$\therefore$ $\frac{\partial M}{\partial y} = 1 + 1/x - \sin y$ and $\frac{\partial N}{\partial x} = 1 + 1/x - \sin y$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $\int \left[y\,(1 + 1/x) + \cos y \right] dx + \int 0\;dy = c$

Thus $\boldsymbol{y\,(x + \log x) + x\cos y = c}$, is the required solution.

>> Let $M = \cos x \tan y + \cos(x + y)$; $N = \sin x \sec^2 y + \cos(x + y)$

$\therefore \quad \dfrac{\partial M}{\partial y} = \cos x \sec^2 y - \sin(x + y), \quad \dfrac{\partial N}{\partial x} = \cos x \sec^2 y - \sin(x + y)$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $\int [\cos x \tan y + \cos(x + y)]\,dx + \int 0\;dy = c$

Thus $\boldsymbol{\sin x \tan y + \sin(x + y) = c}$, is the required solution.

>> The given equation is put in the form,

$$(y\cos x + \sin y + y)\,dx + (\sin x + x\cos y + x)\,dy = 0.$$

Let $M = y\cos x + \sin y + y$ and $N = \sin x + x\cos y + x$

$\therefore \quad \dfrac{\partial M}{\partial y} = \cos x + \cos y + 1$ and $\dfrac{\partial N}{\partial x} = \cos x + \cos y + 1$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $\int (y\cos x + \sin y + y)\,dx + \int 0\,dy = c$

Thus $\boldsymbol{y\sin x + x\sin y + xy = c}$, is the required solution.

32. *Solve* : $(2xy + y - \tan y)\,dx + (x^2 - x\tan^2 y + \sec^2 y)\,dy = 0$

>> Let $M = 2xy + y - \tan y$ and $N = x^2 - x\tan^2 y + \sec^2 y$

$\therefore \quad \dfrac{\partial M}{\partial y} = 2x + 1 - \sec^2 y$ and $\dfrac{\partial N}{\partial x} = 2x - \tan^2 y$

But $\dfrac{\partial M}{\partial y} = 2x + 1 - (1 + \tan^2 y) = 2x - \tan^2 y$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $\int (2xy + y - \tan y)\,dx + \int \sec^2 y\,dy = c$

Thus $\boldsymbol{x^2 y + xy - x\tan y + \tan y = c}$, is the required solution.

---

33. *Solve* : $ye^{xy}\,dx + (xe^{xy} + 2y)\,dy = 0$

>> Let $M = ye^{xy}$, $N = xe^{xy} + 2y$

$\dfrac{\partial M}{\partial y} = ye^{xy}x + e^{xy}$; $\dfrac{\partial N}{\partial x} = xe^{xy}y + e^{xy}$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

Solution is given by $\int M\,dx + \int N(y)\,dy = c$

ie., $\int ye^{xy}\,dx + \int 2y\,dy = c$

ie., $y\dfrac{e^{xy}}{y} + y^2 = c$

Thus $\boldsymbol{e^{xy} + y^2 = c}$, is the required solution.

---

34. *Solve* : $\left(y^2 e^{xy^2} + 4x^3\right)dx + \left(2xy\,e^{xy^2} - 3y^2\right)dy = 0$

>> Let $M = y^2 e^{xy^2} + 4x^3$ and $N = 2xy\,e^{xy^2} - 3y^2$

$\therefore \quad \dfrac{\partial M}{\partial y} = y^2 \cdot e^{xy^2} \cdot 2xy + e^{xy^2} \cdot 2y$, $\dfrac{\partial N}{\partial x} = 2xy\,e^{xy^2} \cdot y^2 + e^{xy^2} \cdot 2y$

ie., $\dfrac{\partial M}{\partial y} = 2xy^3 e^{xy^2} + 2y\,e^{xy^2}$ and $\dfrac{\partial N}{\partial x} = 2xy^3 e^{xy^2} + 2y\,e^{xy^2}$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $\int \left(y^2 e^{xy^2} + 4x^3 \right) dx + \int -3y^2\,dy = c$

ie., $y^2 \cdot \frac{e^{xy^2}}{y^2} + x^4 - y^3 = c$

Thus $e^{xy^2} + x^4 - y^3 = c$, is the required solution.

>> Let $M = 1 + e^{x/y}$ and $N = e^{x/y}\left(1 - \frac{x}{y}\right)$

$\therefore \quad \frac{\partial M}{\partial y} = e^{x/y}\left(-\frac{x}{y^2}\right), \quad \frac{\partial N}{\partial x} = e^{x/y}\cdot\left(-\frac{1}{y}\right) + \left(1 - \frac{x}{y}\right)e^{x/y}\cdot\frac{1}{y}$

ie., $\frac{\partial M}{\partial y} = -\frac{xe^{x/y}}{y^2}, \quad \frac{\partial N}{\partial x} = -\frac{1}{y}e^{x/y} + \frac{1}{y}e^{x/y} - \frac{x}{y^2}e^{x/y} = -\frac{xe^{x/y}}{y^2}$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, **the given equation is exact.**

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $\int \left(1 + e^{x/y}\right) dx + \int 0\,dy = c$

ie., $x + \frac{e^{x/y}}{(1/y)} = c$

Thus $x + y\,e^{x/y} = c$, is the required solution.

Remark : *Refer to Problem - 15, where this Problem is worked by putting* $x = vy$.

36. *Solve :* $(2 + 2x^2\sqrt{y})\,y\,dx + (x^2\sqrt{y} + 2)\,x\,dy = 0$

>> Let $M = 2y + 2x^2 y^{3/2}$ and $N = x^3\sqrt{y} + 2x$

$\therefore \quad \dfrac{\partial M}{\partial y} = 2 + 2x^2 \cdot \dfrac{3}{2} y^{1/2}$ and $\dfrac{\partial N}{\partial x} = 3x^2\sqrt{y} + 2$

ie., $\quad \dfrac{\partial M}{\partial y} = 2 + 3x^2\sqrt{y}$ and $\dfrac{\partial N}{\partial x} = 3x^2\sqrt{y} + 2$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $\quad \int\left(2y + 2x^2 y^{3/2}\right)dx + \int 0\,dy = c$

Thus $\quad 2xy + \dfrac{2}{3}x^3 y^{3/2} = c$, is the required solution.

---

37. *Solve :* $\left(x - \dfrac{y}{x^2+y^2}\right)dx + \left(y + \dfrac{x}{x^2+y^2}\right)dy = 0$ *given that* $y = 1$ *when* $x = 1$.

>> Let $M = x - \dfrac{y}{x^2+y^2}$ and $N = y + \dfrac{x}{x^2+y^2}$

$\therefore \quad \dfrac{\partial M}{\partial y} = \dfrac{(x^2+y^2)(-1) + y \cdot 2y}{(x^2+y^2)^2}, \quad \dfrac{\partial N}{\partial x} = \dfrac{(x^2+y^2)\cdot 1 - x \cdot 2x}{(x^2+y^2)^2}$

ie., $\quad \dfrac{\partial M}{\partial y} = \dfrac{-x^2+y^2}{(x^2+y^2)^2}$ and $\dfrac{\partial N}{\partial x} = \dfrac{-x^2+y^2}{(x^2+y^2)^2}$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie, $\quad \int\left(x - \dfrac{y}{x^2+y^2}\right)dx + \int y\,dy = c$

ie., $\quad \dfrac{x^2}{2} - y \cdot \dfrac{1}{y}\tan^{-1}(x/y) + \dfrac{y^2}{2} = c$

ie., $\quad \dfrac{x^2}{2} + \dfrac{y^2}{2} - \tan^{-1}(x/y) = c$, is the general solution.

Given that $y = 1$ when $x = 1$, the general solution becomes

$$\frac{1}{2}+\frac{1}{2}-\tan^{-1}(1)=c \quad \text{or} \quad c=1-(\pi/4)$$

Thus $\dfrac{x^2}{2}+\dfrac{y^2}{2}-\tan^{-1}(x/y)=1-(\pi/4)$, is the particular solution.

6.35 Equations reducible to the exact form

Sometimes the given differential equation which is not an exact equation can be transformed into an exact equation by multiplying with some function (*factor*) known as the integrating factor (I.F).

The procedure to find such a factor is as follows.

Integrating Factor : Type - 1

Suppose that, for the equation $M\,dx+N\,dy=0$

$\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$, then we take their difference

The difference $\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}$ should be close to the expression of M or N.

If it is so, then we compute $\dfrac{1}{M}\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right)$ or $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right)$

If $\quad \dfrac{1}{N}\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right)=f(x) \quad$ or $\quad \dfrac{1}{M}\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right)=g(y)$

then $e^{\int f(x)\,dx}$ or $e^{-\int g(y)\,dy}$ is an integrating factor.

As already stated the multiplication of this factor into the equation $M\,dx+N\,dy=0$ will make the equation an exact one and we solve this modified equation being an exact equation.

The following basic results will be useful :

(i) $e^{\log x}=x$ (ii) $e^{n\log x}=x^n$

Remark : *If the given equation is not exact we continue the working and try to find out whether it can be reduced to an exact equation as per the described procedure. Even in the case of homogeneous equations it is worth while trying to see whether the equation can be reduced to an exact form.*

[*The first two methods can be ruled out at once for solving this problem*]

>> Let $M = 4xy + 3y^2 - x$ and $N = x(x + 2y) = x^2 + 2xy$

$$\frac{\partial M}{\partial y} = 4x + 6y \text{ and } \frac{\partial N}{\partial x} = 2x + 2y. \text{ (The equation is not exact)}$$

Consider $\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = 2x + 4y = 2(x + 2y)$... close to N.

Now $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = \dfrac{2(x+2y)}{x(x+2y)} = \dfrac{2}{x} = f(x)$

Hence $e^{\int f(x)\,dx}$ is an integrating factor.

ie., $e^{\int f(x)\,dx} = e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log(x^2)} = x^2$

Multiplying the given equation by x^2 we now have,

$$M = 4x^3y + 3x^2y^2 - x^3 \text{ and } N = x^4 + 2x^3y$$

$$\frac{\partial M}{\partial y} = 4x^3 + 6x^2y \text{ and } \frac{\partial N}{\partial x} = 4x^3 + 6x^2y$$

Remark : *Checking the exactness condition as such is not required as the given equation is sure to reduce to an exact equation on multiplication with the proper I.F. But it is just a safety check as it will give an opportunity for one to rectify the mistake in the event of the exactness condition not being satisfied.*

Solution of the exact equation is $\int M\,dx + \int N(y)\,dy = c$

ie., $\int\left(4x^3y + 3x^2y^2 - x^3\right)dx + \int 0\,dy = c$

Thus $x^4y + x^3y^2 - \dfrac{x^4}{4} = c$, is the required solution.

>> Let $M = y(2x - y + 1)$ and $N = x(3x - 4y + 3)$

ie., $M = 2xy - y^2 + y$ and $N = 3x^2 - 4xy + 3x$

$$\frac{\partial M}{\partial y} = 2x - 2y + 1, \quad \frac{\partial N}{\partial x} = 6x - 4y + 3$$

$\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = -4x + 2y - 2 = -2(2x - y + 1)$... near to M.

Now, $\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{-2(2x-y+1)}{y(2x-y+1)}=-\frac{2}{y}=g(y)$

Hence $I\cdot F=e^{-\int g(y)\,dy}=e^{\int \frac{2}{y}dy}=e^{2\log y}=e^{\log(y^2)}=y^2$

Multiplying the given equation with y^2 we now have,

$$M=2xy^3-y^4+y^3 \quad \text{and} \quad N=3x^2y^2-4xy^3+3xy^2$$

$$\frac{\partial M}{\partial y}=6xy^2-4y^3+3y^2 \quad \text{and} \quad \frac{\partial N}{\partial x}=6xy^2-4y^3+3y^2$$

The solution is $\int M\,dx+\int N(y)\,dy=c$

ie., $\int\left(2xy^3-y^4+y^3\right)dx+\int 0\,dy=c$

Thus $x^2y^3-xy^4+xy^3=c$, is the required solution.

>> Let $M=x^2+y^2+x$ and $N=xy$

$$\frac{\partial M}{\partial y}=2y \quad \text{and} \quad \frac{\partial N}{\partial x}=y$$

$$\therefore \quad \frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=2y-y=y\ldots \text{near to } N.$$

Now $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{y}{xy}=\frac{1}{x}=f(x)$

Hence $I\cdot F=e^{\int f(x)\,dx}=e^{\int \frac{1}{x}dx}=e^{\log x}=x$

Multiplying the given equation by x, we now have,

$$M=x^3+xy^2+x^2 \quad \text{and} \quad N=x^2y$$

$$\frac{\partial M}{\partial y}=2xy \quad \text{and} \quad \frac{\partial N}{\partial x}=2xy$$

The solution is $\int M\,dx+\int N(y)\,dy=c$

ie., $\int\left(x^3+xy^2+x^2\right)dx+\int 0\,dy=c$

Thus $\frac{x^4}{4}+\frac{x^2y^2}{2}+\frac{x^3}{3}=c$, is the required solution.

41. *Solve :* $y(2xy+1)\,dx - x\,dx = 0$

>> Let $M = y(2xy+1)$ and $N = -x$

$$\frac{\partial M}{\partial y} = 4xy+1 \qquad \text{and} \quad \frac{\partial N}{\partial x} = -1$$

$$\therefore \quad \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4xy+2 = 2(2xy+1)\ldots \text{near to } M.$$

Now, $\dfrac{1}{M}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = \dfrac{2(2xy+1)}{y(2xy+1)} = \dfrac{2}{y} = g(y)$

Hence $I \cdot F = e^{-\int g(y)\,dy} = e^{-\int \frac{2}{y}dy} = e^{-2\log y} = y^{-2} = \dfrac{1}{y^2}$

Multiplying the given equation with $1/y^2$, we now have

$$M = 2x + \frac{1}{y} \quad \text{and} \quad N = -\frac{x}{y^2}$$

$$\frac{\partial M}{\partial y} = -\frac{1}{y^2} \quad \text{and} \quad \frac{\partial N}{\partial x} = -\frac{1}{y^2}$$

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $\quad \int\left(2x + \dfrac{1}{y}\right)dx + \int 0\,dy = c$

Thus $\quad x^2 + \dfrac{x}{y} = c$, is the required solution.

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42. *Solve :* $y(x+y)\,dx + (x+2y-1)\,dy = 0$

>> Let $M = xy + y^2$ and $N = x + 2y - 1$

$$\frac{\partial M}{\partial y} = x + 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = 1$$

$$\therefore \quad \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = x + 2y - 1\ldots \text{near to } N.$$

Now $\quad \dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = \dfrac{x+2y-1}{x+2y-1} = 1 = f(x)$

[*Though the constant 1 can be regarded as a function of x or y, we take it to be $f(x)$ since the difference in the partial derivatives is divided by N*]

Hence $I \cdot F = e^{\int f(x)\,dx} = e^{\int 1\,dx} = e^x$

Multiplying the given equation by e^x, we now have

$$M = e^x\,(xy + y^2) \quad \text{and} \quad N = e^x\,(x + 2y - 1)$$

$$\frac{\partial M}{\partial y} = e^x x + e^x \cdot 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = e^x\,(x + 2y - 1) + e^x$$

ie., $$\frac{\partial M}{\partial y} = x e^x + 2 e^x y \quad \text{and} \quad \frac{\partial N}{\partial x} = x e^x + 2 e^x y$$

The solution is $\int M\,dx + \int N\,(y)\,dy = c$

ie., $$y \int x e^x\,dx + y^2 \int e^x\,dx + \int 0\,dy = c$$

ie., $$y\,(x e^x - e^x) + y^2 e^x = c$$

Thus $\boldsymbol{e^x\,(xy + y^2 - y) = c}$, is the required solution.

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43. Solve : $(8xy - 9y^2)\,dx + 2\,(x^2 - 3xy)\,dy = 0$

[*It may be observed that the given equation is a homogeneous one and before one ventures to use the substitution $y = vx$, the option of exact equation is worth while trying* .]

>> Let $M = 8xy - 9y^2$ and $N = 2x^2 - 6xy$

$$\frac{\partial M}{\partial y} = 8x - 18y \quad \text{and} \quad \frac{\partial N}{\partial x} = 4x - 6y$$

$\therefore$ $\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = 4x - 12y = 4\,(x - 3y)$ is near to $N = 2x\,(x - 3y)$

Now $$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{4\,(x - 3y)}{2x\,(x - 3y)} = \frac{2}{x} = f(x)$$

Hence $$I \cdot F = e^{\int f(x)\,dx} = e^{\int \frac{2}{x}\,dx} = e^{2 \log x} = x^2$$

Multiplying the given equation by x^2, we now have

$$M = 8x^3 y - 9x^2 y^2 \quad \text{and} \quad N = 2\,x^4 - 6\,x^3 y$$

$$\frac{\partial M}{\partial y} = 8x^3 - 18x^2 y \quad \text{and} \quad \frac{\partial N}{\partial x} = 8x^3 - 18x^2 y$$

The solution is $\int M\,dx + \int N\,(y)\,dy = c$

ie., $$\int \left(8\,x^3 y - 9x^2 y^2\right) dx + \int 0\,dy = c$$

ie., $\boldsymbol{2\,x^4 y - 3\,x^3 y^2 = c}$, is the required solution.

If the given equation $M\,dx + N\,dy = 0$ is of the form

$$y f(xy)\,dx + x\,g(xy)\,dy = 0$$

then $\dfrac{1}{Mx - Ny}$ is an integrating factor provided $Mx - Ny \neq 0$

Remark : *It should be clearly noted that 'f' and 'g' are function of (xy) not in the variables x, y*

>> The given equation is of the form :

$$y f(xy)\,dx + x\,g(xy)\,dy = 0 \text{ where}$$

$$M = y f(xy) = xy^2 + y \text{ and } N = x\,g(xy) = x - x^2 y$$

Now $Mx - Ny = x^2 y^2 + xy - xy + x^2 y^2 = 2x^2 y^2$

$\therefore \quad \dfrac{1}{Mx - Ny} = \dfrac{1}{2x^2 y^2}$ is the integrating factor.

Multiplying the given equation with $1/2x^2 y^2$ it becomes an exact equation where we now have,

$$M = \frac{1}{2x^2 y^2}(xy^2 + y) \text{ and } N = \frac{1}{2x^2 y^2}(x - x^2 y)$$

ie., $\quad M = \dfrac{1}{2x} + \dfrac{1}{2x^2 y} \quad$ and $\quad N = \dfrac{1}{2xy^2} - \dfrac{1}{2y}$

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $\quad \displaystyle\int \left(\frac{1}{2x} + \frac{1}{2x^2 y} \right) dx + \int -\frac{1}{2y}\,dy = c$

ie., $\quad \dfrac{1}{2}\log x - \dfrac{1}{2xy} - \dfrac{1}{2}\log y = c$

or $\quad \log(x/y) - \dfrac{1}{xy} = 2c$, is the required solution.

>> The equation is of the form $y f(xy)\,dx + x\,g(xy)\,dy = 0$ where,

$$M = y f(xy) = y + xy^2 + x^2 y^3 \text{ and}$$

$$N = x\,g(xy) = x - x^2 y + x^3 y^2$$

Now $Mx - Ny = \left(xy + x^2 y^2 + x^3 y^3\right) - \left(xy - x^2 y^2 + x^3 y^3\right) = 2x^2 y^2$

$$\therefore \quad \frac{1}{Mx - Ny} = \frac{1}{2x^2 y^2} \text{ is the I.F.}$$

Multiplying the given equation with $1/2x^2 y^2$ it becomes an exact equation where we now have,

$$M = \frac{1}{2x^2 y} + \frac{1}{2x} + \frac{y}{2} \quad \text{and} \quad N = \frac{1}{2xy^2} - \frac{1}{2y} + \frac{x}{2}$$

The solution is given by $\int M\,dx + \int N(y)\,dy = c$

ie., $$\int \left(\frac{1}{2x^2 y} + \frac{1}{2x} + \frac{y}{2} \right) dx + \int -\frac{1}{2y}\,dy = c$$

ie., $$-\frac{1}{2xy} + \frac{1}{2}\log x + \frac{xy}{2} - \frac{1}{2}\log y = c$$

or $$\log(x/y) + xy - \frac{1}{xy} = 2c \text{ , is the required solution.}$$

>> The equation of the form $y f(xy)\,dx + x\,g(xy)\,dy = 0$ where,

$$M = xy^2 + 2x^2 y^3 \quad \text{and} \quad N = x^2 y - x^3 y^2$$

Now $Mx - Ny = x^2 y^2 + 2x^3 y^3 - x^2 y^2 + x^3 y^3 = 3x^3 y^3$

Thus $1/3x^3 y^3$ is the I.F. Multiplying the given equation by this I.F we have an exact equation where we now have,

$$M = \frac{1}{3x^2 y} + \frac{2}{3x} \quad \text{and} \quad N = \frac{1}{3xy^2} - \frac{1}{3y}$$

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $$\int \left(\frac{1}{3x^2 y} + \frac{2}{3x} \right) dx + \int -\frac{1}{3y}\,dy = c$$

ie., $$-\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c$$

or $\quad -\dfrac{1}{xy} + \log x^2 - \log y = 3c$

Thus $\quad \mathbf{\log(x^2/y) - 1/xy = 3c}$, is the required solution.

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>> The equation is of the form $y f(xy)\,dx + x\,g(xy)\,dy = 0$

$$M = xy^2 \sin(xy) + y\cos(xy) \quad \text{and} \quad N = x^2 y \sin(xy) - x\cos(xy)$$

Now $\quad Mx - Ny = 2xy\cos(xy)$

Thus $\quad \dfrac{1}{2xy\cos(xy)} = \dfrac{\sec(xy)}{2xy}$ is the I.F.

Multiplying the given equation with this I.F we have,

$$M = \frac{y}{2}\tan(xy) + \frac{1}{2x} \quad \text{and} \quad N = \frac{x}{2}\tan(xy) - \frac{1}{2y}$$

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $\quad \int\left[\dfrac{y}{2}\tan(xy) + \dfrac{1}{2x}\right]dx + \int -\dfrac{1}{2y}\,dy = c$

ie., $\quad \dfrac{y}{2}\,\dfrac{\log[\sec(xy)]}{y} + \dfrac{1}{2}\log x - \dfrac{1}{2}\log y = c$

or $\quad \log\sec(xy) + \log(x/y) = 2c$

or $\quad \log\left[\sec(xy)\cdot(x/y)\right] = \log k \quad$ (*say*)

$\Rightarrow \quad \sec(xy)\cdot(x/y) = k$

Thus $\mathbf{x\sec(xy) = ky}$, is the required solution.

Integrating factor [illegible]

If the given equation $M\,dx + N\,dy = 0$ is of the form

$$x^{k_1} y^{k_2}(c_1 y\,dx + c_2 x\,dy) + x^{k_3} y^{k_4}(c_3 y\,dx + c_4 x\,dy) = 0$$

where k_i and c_i $(i = 1 \text{ to } 4)$ are constants then $x^a y^b$ is an integrating factor. The constants a and b are determined such that the condition for an exact equation is satisfied.

Remark : *It may be observed that in this type, the terms M and N of the equations are of the form $x^a\ y^b$ only where a and b are constants. In such a case we multiply the equation* *simply by* $x^a\ y^b$ *and find* $a,\ b$ *such that* $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. *Later we solve the equation by using the values of* a *and* b *so obtained.*

>> The given equation can be rearranged as

$$(3xy + 8y^5)\,dx + (2x^2 + 24xy^4)\,dy = 0 \qquad \ldots (1)$$

[*This equation is not an exact equation and the first two types of finding the I.F can be ruled out easily. Observe that the equation contains only terms of the form* $x^a\ y^b$]

Multiplying (1) by $x^a y^b$ we have

$$M = 3\,x^{a+1}\,y^{b+1} + 8\,x^a\,y^{b+5}$$

$$N = 2\,x^{a+2}\,y^b + 24\,x^{a+1}\,y^{b+4}$$

We shall find a and b such that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore \quad 3x^{a+1}(b+1)\,y^b + 8x^a(b+5)\,y^{b+4} = 2(a+2)\,x^{a+1}\,y^b + 24(a+1)\,x^a\,y^{b+4}$

$\Rightarrow \quad 3(b+1) = 2(a+2)$ and $8(b+5) = 24(a+1)$

or $\quad 2a - 3b = -1$ and $3a - b = 2$

By solving we get $a = 1,\ b = 1$.

We now have $M = 3x^2\,y^2 + 8xy^6$ and $N = 2x^3\,y + 24x^2\,y^5$

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $\quad \int \left(3x^2 y^2 + 8xy^6\right) dx + \int 0\ dy = c$

Thus $\quad x^3\,y^2 + 4\,x^2\,y^6 = c$, is the required solution.

>> We have $(4xy + 3y^4)\,dx + (2x^2 + 5xy^3)\,dy = 0$

Multiplying the equation by $x^a\,y^b$ we have,

$$M = 4x^{a+1}\,y^{b+1} + 3x^a\,y^{b+4} \text{ and } N = 2x^{a+2}\,y^b + 5x^{a+1}\,y^{b+3}$$

$$\frac{\partial M}{\partial y} = 4(b+1)\,x^{a+1}\,y^b + 3(b+4)\,x^a\,y^{b+3}$$

$$\frac{\partial N}{\partial x} = 2(a+2)x^{a+1}y^b + 5(a+1)x^a y^{b+3}$$

We have to find a and b such that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow \quad 4(b+1) = 2(a+2)$ and $3(b+4) = 5(a+1)$

ie., $\quad a = 2b$ and $5a - 3b = 7$

By solving we get $a = 2$ and $b = 1$

We now have, $M = 4x^3y^2 + 3x^2y^5$ and $N = 2x^4y + 5x^3y^4$

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $\quad \int\left(4x^3y^2 + 3x^2y^5\right)dx + \int 0\,dy = c$

Thus $\quad x^4y^2 + x^3y^5 = c$, is the required solution.

[*We cannot find I. F by the earlier described methods*]

>> Multiplying the given equation by $x^a y^b$ we have

$$M = x^a y^{b+2} + 2x^{a+2}y^{b+1} \text{ and } N = 2x^{a+3}y^b - x^{a+1}y^{b+1}$$

$$\frac{\partial M}{\partial y} = (b+2)x^a y^{b+1} + 2(b+1)x^{a+2}y^b$$

$$\frac{\partial N}{\partial x} = 2(a+3)x^{a+2}y^b - (a+1)x^a y^{b+1}$$

Let us find a and b such that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow \quad (b+2) = -(a+1)$ and $2(b+1) = 2(a+3)$

ie., $\quad a+b = -3$ and $a-b = -2$

By solving we get $a = -5/2$ and $b = -1/2$.

We now have,

$$M = x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2} \text{ and } N = 2x^{1/2}y^{-1/2} - x^{-3/2}y^{1/2}$$

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $\quad \int\left(x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2}\right)dx + \int 0\,dy = c$

ie., $$\frac{x^{-3/2}}{-3/2}y^{3/2} + 2\frac{x^{1/2}}{1/2}y^{1/2} = c$$

ie., $$\frac{-2}{3}x^{-3/2}y^{3/2} + 4x^{1/2}y^{1/2} = c$$

or $$-x^{-3/2}y^{3/2} + 6x^{1/2}y^{1/2} = 3c/2$$

Thus $6\sqrt{xy} - \sqrt{y^3/x^3} = k$, is the required solution, where $k = 3c/2$

The preamble for this method is that we should be in a position to recognize certain *standard exact differentials* listed as follows.

E_1. $dx \pm dy = d(x \pm y)$

E_2. $x\,dy + y\,dx = d(xy)$

E_3. $\dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$

E_4. $\dfrac{y\,dx - x\,dy}{y^2} = d\left(\dfrac{x}{y}\right)$

E_5. $\dfrac{x\,dy - y\,dx}{x^2 - y^2} = d\left[\dfrac{1}{2}\log\left(\dfrac{x+y}{x-y}\right)\right]$

E_6. $\dfrac{x\,dy - y\,dx}{x^2 + y^2} = d\left[\tan^{-1}\left(\dfrac{y}{x}\right)\right] = -d\left[\tan^{-1}\left(\dfrac{x}{y}\right)\right]$

E_7. $\dfrac{x\,dx + y\,dy}{x^2 + y^2} = d\left[\dfrac{1}{2}\log(x^2 + y^2)\right] = d\left[\log\sqrt{x^2 + y^2}\right]$

E_8. $\dfrac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}} = d\left[\sqrt{x^2 + y^2}\right]$

With this preamble we describe the method.

The given equation itself can be a combination of various standard exact differentials or we may have to judiciously rearrange the terms of the given equation to match with the exact differentials fully or partly as the case may be. In otherwords the given equation must assume the form

$$c_1\,d[f_1(x, y)] + c_2\,d[f_2(x, y)] + \cdots = 0$$

In such a case, the solution can instantly be written as

$$c_1 f_1(x, y) + c_2 f_2(x, y) + \cdots = c, \text{ by integration.}$$

>> The given equation is equivalent to the form

$$x\,dx + y\,dy - d\,[\,\tan^{-1}(y/x)\,]$$

Integrating we get,

$$\frac{x^2}{2} + \frac{y^2}{2} - \tan^{-1}(y/x) = c,$$ being the required solution.

Remark : *Had we not recognized the third term of the given equation as* $d\,[\,\tan^{-1}(y/x)\,]$ *then in the normal course we should have written the given equation in the form*

$$\left[x + \frac{y}{x^2+y^2}\right]dx + \left[y - \frac{x}{x^2+y^2}\right]dy = 0$$

and solve by verifying the exactness condition similar to Problem - 37.Further since $d\left[\tan^{-1}(y/x)\right] = -d\left[\tan^{-1}(x/y)\right]$ the solution can also be in the form

$$\frac{x^2}{2} + \frac{y^2}{2} + \tan^{-1}(x/y) = c$$

>> The given equation is equivalent to the form,

$$d\left(\frac{x}{y}\right) + x\,dx + y\,dy = 0$$

$\Rightarrow$ $\dfrac{x}{y} + \dfrac{x^2}{2} + \dfrac{y^2}{2} = c$, on integration.

Thus $\dfrac{x}{y} + \dfrac{1}{2}(x^2+y^2) = c$, is the required solution.

[It should be carefully noted that though we have (x/y) terms in the equation it is not a homogeneous equation]

>> The given equation can be put in the form,

$$e^{x/y}(y\,dx - x\,dy) = y^2\,dy$$

or $\quad e^{x/y}\left[\dfrac{y\,dx - x\,dy}{y^2}\right] = dy$

ie., $\quad e^{x/y}\,d\,(x/y) = dy$

Integration yields $e^{x/y} = y + c$

Thus $\quad \boldsymbol{e^{x/y} - y = c}$, is the required solution.

Remark : *The Problem can be solved by verifying the exactness condition and it will be little difficult.*

\>> The given equation can be put in the form

$$x\,dx + y\,dy = y\,(x^2 + y^2)\,dy \quad \text{or} \quad \frac{x\,dx + y\,dy}{x^2 + y^2} = y\,dy$$

ie., $\quad d\left[\dfrac{1}{2}\log(x^2 + y^2)\right] = y\,dy$

Integrating we get, $\dfrac{1}{2}\log(x^2 + y^2) = \dfrac{y^2}{2} + c$

or $\quad \log(x^2 + y^2) = y^2 + 2c$ and let $k = 2c$

Thus $\quad \boldsymbol{\log(x^2 + y^2) - y^2 = k}$, is the required solution.

\>> The given equation can be put in the form

$$dx + \tan(xy)\,[\,y\,dx + x\,dy\,] = 0$$

ie., $\quad dx + \tan(xy)\,d\,(xy) = 0$

Integrating we get, $\boldsymbol{x + \log\sec(xy) = c}$, being the required solution.

Solve the following differential equations

1. $3x(xy-2)\,dx + (x^3+2y)\,dy = 0$

2. $(\cos 2y - 3x^2y^2)\,dx + (\cos 2y - 2x\sin 2y - 2x^3y)\,dy = 0$

3. $(6x^2 + 2xy - 2xye^{-x^2})\,dx + (e^{-x^2} + x^2 + 3y^2)\,dy = 0$

4. $(2x^2 + 6xy - y^2)\,dx + (3x^2 - 2xy + y^2)\,dy = 0$

5. $\cos x\,(e^y + 1)\,dx + \sin x\, e^y\,dy = 0$ *[solve by two methods]*

6. $\left[4x^3y^2 + y\cos(xy)\right]dx + \left[2x^4y + x\cos(xy)\right]dy = 0$

7. $\dfrac{dy}{dx} = \dfrac{y+x-2}{y-x-4}$ *[solve by two methods]*

8. $(xy^2 + x - 2y + 3)\,dx + x^2y\,dy = 2(x+y)\,dy\ ;\ y(1) = 1$

9. $y(x+y+1)\,dx + x(x+3y+2)\,dy = 0$

10. $2(3x^2 + 2y^3 + 6y)\,dx + 3(x + xy^2)\,dy = 0$ given that $y(1) = 2$

11. $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$

12. $(x^7y^2 + 3y)\,dx + (3x^8y - x)\,dy = 0$

13. $(x^2y^2 + 5xy + 2)\,y\,dx + (x^2y^2 + 4xy + 2)\,x\,dy = 0$

14. $y\,dx - x\,dy + 3x^2y^2e^{x^3}\,dx = 0$

15. $x\,dy - y\,dx = \cos^2(y/x)\,dx$

1. $x^3y - 3x^2 + y^2 = c$	2. $\dfrac{\sin 2y}{2} + x\cos 2y - x^3y^2 = c$
3. $2x^3 + x^2y + ye^{-x^2} + y^3 = c$	4. $2x^3 + 9x^2y - 3xy^2 + y^3 = k$
5. $\sin x\,(e^y + 1) = c$	6. $x^4y^2 + \sin(xy) = c$
7. $x^2 + 2xy - y^2 - 4x + 8y = c$	8. $\dfrac{x^2y^2}{2} + \dfrac{x^2}{2} - 2xy + 3x - y^2 = 1$
9. $\dfrac{x^2y^2}{2} + xy^3 + xy^2 = c$	10. $x^6 + x^4y^3 + 3x^4y = 15$
11. $x^3y^3 + x^2 = cy$	12. $2x^7y^3 - y^2 = cx^6$

13. $\log(x^5/y^4) = (2/xy) - xy + c$

14. $(x/y) + e^{x^3} = c$

15. $\tan(y/x) + 1/x = c$

[illegible] Linear Differential [illegible]

A differential equation is said to be *linear* if the dependent variable and its derivative occurs in the first degree only and they are not multiplied together.

[illegible] form of a lin[illegible]

A differential equation of the form

$$\frac{dy}{dx} + Py = Q \qquad \ldots (1)$$

where P and Q are functions of x only is called a *linear equation in 'y'* and we shall solve the same.

Multiplying (1) by $e^{\int P\,dx}$ we have,

$$e^{\int P\,dx}\frac{dy}{dx} + P\,y\,e^{\int P\,dx} = Q\,e^{\int P\,dx} \quad \text{or} \quad \frac{d}{dx}\left[e^{\int P\,dx}\,y\right] = Q\,e^{\int P\,dx}$$

Integrating w.r.t. x on both sides we have,

$$e^{\int P\,dx}\,y = \int Q\,e^{\int P\,dx}\,dx + c$$

Thus $y\,e^{\int P\,dx} = \int Q e^{\int P\,dx}\,dx + c$ is the solution of linear equation (1).

An equation of the form : $\frac{dx}{dy} + P\,x = Q$ where P and Q are functions of y only is called a *linear equation in x*.

The solution can simply be written by interchanging the role of x and y in the solution obtained already for the linear equation in y.

ie., $x\,e^{\int P\,dy} = \int Q\,e^{\int P\,dy}\,dy + c$

This is the solution for the linear equation in x.

- The given equation must be first put in the form conformal to the standard form of the linear equation in x or y.
- The expression for P and Q is to be written by simple comparison.

- We equip with the I. F $e^{\int P\,dx}$ or $e^{\int P\,dy}$.
- We assume the associated solution and we only need to tackle the R.H.S part of the solution to finally arrive at the required solution.

Note : *If the given equation is not in the standard form of a linear equation but is in the form $M(x, y)\,dx + N(x, y)\,dy = 0$ then the linearity of the D. E can be recognized by the following observations.*

1. ***Linear in y*** : *M contains 'y' (not y^2,* log *y* etc.), *N is a function of x only.*
2. ***Linear in x*** : *N contains 'x'* (not x^2, sin *x* etc), *M is a function of y* only.

In such a case the equation is transformed into the standard form of the linear equation and can be solved by assuming the solution.

Remark : *Many of the problems when the equation is given in the form $M(x, y)\,dx + N(x, y)\,dy = 0$ can be solved by the earlier methods. If we are able to recognize the linearity then we can assume the solution in respect of equations in the standard form of linear equation.*

WORKED PROBLEMS

56. Solve $\frac{dy}{dx} - \frac{2y}{x} = x + x^2$

>> $\frac{dy}{dx} - \frac{2y}{x} = x + x^2$ is of the form :

$$\frac{dy}{dx} + Py = Q, \text{ where } P = \frac{-2}{x} \text{ and } Q = x + x^2$$

$$\therefore \quad e^{\int P\,dx} = e^{-\int 2/x\,dx} = e^{-2\log x} = e^{\log x^{-2}} = 1/x^2$$

The solution is given by $y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$

ie., $$y \cdot \frac{1}{x^2} = \int (x + x^2)\frac{1}{x^2}\,dx + c$$

ie., $$\frac{y}{x^2} = \int \frac{1}{x}\,dx + \int 1\,dx + c$$

Thus $\frac{y}{x^2} = \log x + x + c$, is the required solution.

57. Solve: $\frac{dy}{dx} + y\cot x = \cos x$

>> $\frac{dy}{dx} + y\cot x = \cos x$ is of the form :

$$\frac{dy}{dx} + Py = Q, \text{ where } P = \cot x \text{ and } Q = \cos x$$

$\therefore \quad e^{\int P\,dx} = e^{\int \cot x\,dx} = e^{\log(\sin x)} = \sin x$

The solution is $y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$

ie., $\quad y\sin x = \int \cos x \cdot \sin x\,dx + c$

ie., $\quad y\sin x = \int \frac{\sin 2x}{2}\,dx + c$

Thus $\quad \boldsymbol{y\sin x = \frac{-\cos 2x}{4} + c}$, is the required solution.

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58. Solve: $x\cos x\frac{dy}{dx} + (\cos x - x\sin x)y = 1$

>> Dividing the given equation through out by $x\cos x$ we have,

$$\frac{dy}{dx} + \left[\frac{\cos x - x\sin x}{x\cos x}\right]y = \frac{1}{x\cos x}$$

This is of the form $\frac{dy}{dx} + Py = Q$ where

$$P = \frac{\cos x - x\sin x}{x\cos x} \text{ and } Q = \frac{1}{x\cos x}$$

$\therefore \quad e^{\int P\,dx} = e^{\log(x\cos x)} = x\cos x$

The solution is $y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$

ie., $\quad y \cdot x\cos x = \int \frac{1}{x\cos x} \cdot x\cos x\,dx + c$

Thus $\boldsymbol{x\,y\cos x = x + c}$, is the required solution.

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59. *Solve :* $2y'$ [illegible]

>> Dividing the given equation by 2 cos x, we have

$$\frac{dy}{dx} + (2\tan x)\,y = \sin x \quad \because \quad \sin 2x = 2\cos x \sin x$$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = 2\tan x$ and $Q = \sin x$.

$\therefore \qquad e^{\int P dx} = e^{\int 2\tan x\, dx} = e^{2\log(\sec x)} = \sec^2 x$

The solution is $\; y\, e^{\int P dx} = \int Q\, e^{\int P dx}\, dx + c$

ie., $\qquad y\sec^2 x = \int \sin x \sec^2 x\, dx + c$

ie., $\qquad y\sec^2 x = \int \tan x \sec x\, dx + c$

ie., $\qquad y\sec^2 x = \sec x + c$, is the general solution . . . (1)

Consider $y(\pi/3) = 0$. That is $y = 0$ when $x = \pi/3$

Hence (1) becomes $0 = 2 + c$ or $c = -2$

Thus $\quad$ **$y\sec^2 x = \sec x - 2$**, is the required particular solution.

60. *Solve* [illegible]

>> $\qquad \frac{dy}{dx} + y\cot x = 4x \operatorname{cosec} x$ is of the form :

$$\frac{dy}{dx} + Py = Q, \text{ where } P = \cot x \text{ and } Q = 4x \operatorname{cosec} x$$

Now $\; e^{\int P dx} = e^{\int \cot x\, dx} = e^{\log(\sin x)} = \sin x.$

Solution is given by $\; y\, e^{\int P dx} = \int Q\, e^{\int P dx}\, dx + c$

i.e., $\qquad y\sin x = \int 4x \operatorname{cosec} x \cdot \sin x\, dx + c$

i.e., $\qquad y\sin x = \int 4x\, dx + c$

i.e., $\qquad$ **$y\sin x = 2x^2 + c$** is the general solution.

But $\qquad y = 0$ when $x = \pi/2$ by data.

$\therefore \qquad 0 = \pi^2/2 + c \qquad \therefore \; c = -\pi^2/2$

Thus **$y\sin x = 2x^2 - (\pi^2/2)$**, is the particular solution.

>> The given equation is of the form $\frac{dy}{dx} + Py = Q$

where $P = 3x^2$ and $Q = x^5 e^{x^3}$ $\quad \therefore \; e^{\int P\,dx} = e^{\int 3x^2 dx} = e^{x^3}$

The solution is $y\, e^{\int P\,dx} = \int Q\, e^{\int P\,dx} dx + c$

ie., $\quad y\, e^{x^3} = \int x^5 e^{x^3} \cdot e^{x^3} dx + c$

ie., $\quad y\, e^{x^3} = \int x^5 e^{2x^3} dx + c \qquad \ldots (1)$

Put $\quad x^3 = t \qquad \therefore \; 3x^2 dx = dt$ or $3x^5 dx = x^3 dt$

Hence $x^5 dx = \frac{t\,dt}{3}$

Thus (1) becomes $y\, e^{x^3} = \frac{1}{3} \int t\, e^{2t} dt + c$

ie., $\quad y\, e^{x^3} = \frac{1}{3} \left\{ t \frac{e^{2t}}{2} - \int \frac{e^{2t}}{2} \cdot 1\, dt \right\} + c,$

ie., $\quad y\, e^{x^3} = \frac{1}{3} \left\{ \frac{te^{2t}}{2} - \frac{e^{2t}}{4} \right\} + c$

ie., $\quad y\, e^{x^3} = \frac{e^{2t}}{12} (2t - 1) + c$

Thus $\quad y\, e^{x^3} = \frac{e^{2x^3}}{12} (2x^3 - 1) + c,$ is the required solution.

>> Dividing the given equation throughout by $x(1 - x^2)$ we have,

$$\frac{dy}{dx} + \left\{ \frac{2x^2 - 1}{x(1 - x^2)} \right\} y = \frac{x^2}{(1 - x^2)}$$

This is of the form $\frac{dy}{dx} + Py = Q$, where

$$P = \frac{2x^2 - 1}{x(1 - x^2)} \quad \text{and} \quad Q = \frac{x^2}{1 - x^2}$$

$\int P\,dx$ has to be found first, by resolving P into partial fractions.

Let $\dfrac{2x^2-1}{x(1-x)(1+x)} = \dfrac{A}{x} + \dfrac{B}{1-x} + \dfrac{C}{1+x}$

$\Rightarrow \quad 2x^2-1 = A(1-x)(1+x) + Bx(1+x) + Cx(1-x)$

Put $x = 0$: $-1 = A(1)$ $\therefore$ $A = -1$

Put $x = 1$: $1 = B(2)$ $\therefore$ $B = 1/2$

Put $x = -1$: $1 = C(-2)$ $\therefore$ $C = -1/2$

$$\int P\,dx = -\int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{1}{1-x}\,dx - \frac{1}{2}\int \frac{1}{1+x}\,dx$$

$$= -\log x - \frac{1}{2}\log(1-x) - \frac{1}{2}\log(1+x)$$

$$= -\left[\log x + \log\sqrt{1-x} + \log\sqrt{1+x}\right]$$

$$= -\log\left(x\sqrt{1-x^2}\right)$$

Hence $e^{\int P\,dx} = e^{-\log[x\sqrt{1-x^2}]} = \dfrac{1}{x\sqrt{1-x^2}}$

The solution is $y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$

ie., $\dfrac{y}{x\sqrt{1-x^2}} = \int \dfrac{x^2}{(1-x^2)} \cdot \dfrac{1}{x\sqrt{1-x^2}}\,dx + c$

ie., $\dfrac{y}{x\sqrt{1-x^2}} = \int \dfrac{x}{(1-x^2)^{3/2}}\,dx + c$

Put $(1-x^2) = t$ $\therefore$ $-2x\,dx = dt$ or $x\,dx = dt/-2$

$\therefore \quad \dfrac{y}{x\sqrt{1-x^2}} = \dfrac{-1}{2}\int \dfrac{dt}{t^{3/2}} + c$

ie., $\dfrac{y}{x\sqrt{1-x^2}} = \dfrac{-1}{2}\cdot\dfrac{t^{-1/2}}{(-1/2)} + c$

ie., $\dfrac{y}{x\sqrt{1-x^2}} = \dfrac{1}{\sqrt{1-x^2}} + c$

or $\quad \boldsymbol{y = x + c\,x\sqrt{1-x^2}}$, is the required solution.

63. *Solve* : $\sqrt{1-y^2}\, dx + (x - \sin^{-1} y)\, dy = 0$

In the given equation, observe that M is a function of y and N contains x (x dy is present in the equation) Hence the equation is linear in x.

$$\gg \qquad \frac{dx}{dy} = \frac{\sin^{-1} y - x}{\sqrt{1-y^2}}$$

or $$\frac{dx}{dy} + \frac{x}{\sqrt{1-y^2}} = \frac{\sin^{-1} y}{\sqrt{1-y^2}}$$ and this is of the form

$$\frac{dx}{dy} + Px = Q, \quad \text{where } P = \frac{1}{\sqrt{1-y^2}} \quad \text{and} \quad Q = \frac{\sin^{-1} y}{\sqrt{1-y^2}}$$

$$\therefore \qquad e^{\int P\, dy} = e^{\sin^{-1} y}$$

The solution is $x\, e^{\int P\, dy} = \int Q\, e^{\int P\, dy}\, dy + c$

ie., $$x\, e^{\sin^{-1} y} = \int \frac{\sin^{-1} y}{\sqrt{1-y^2}}\, e^{\sin^{-1} y}\, dy + c$$

Put $\sin^{-1} y = t \quad \therefore \quad \dfrac{1}{\sqrt{1-y^2}}\, dy = dt$

$$\therefore \qquad x\, e^{\sin^{-1} y} = \int t\, e^t\, dt + c$$

ie., $x\, e^{\sin^{-1} y} = t\, e^t - e^t + c$, on integration by parts.

Thus $x\, e^{\sin^{-1} y} = e^{\sin^{-1} y}\left(\sin^{-1} y - 1\right) + c$, is the required solution.

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64. *Solve* : $(1 + y^2)\, dx + (x - \tan^{-1} y)\, dy = 0$

This problem is very much similar to the previous one. Observing the same features in this equation we recognize the equation as a linear equation in x.

We have $$\frac{dx}{dy} = \frac{\tan^{-1} y - x}{1 + y^2}$$

or $$\frac{dx}{dy} + \frac{x}{1 + y^2} = \frac{\tan^{-1} y}{1 + y^2}$$

This is of the form $\dfrac{dx}{dy} + Px = Q$

Here $P = \dfrac{1}{1 + y^2}$ and $Q = \dfrac{\tan^{-1} y}{1 + y^2}$ and $e^{\int P\, dy} = e^{\tan^{-1} y}$

The solution is given by $x\,e^{\int P dy} = \int Q\,e^{\int P dy}\,dy + c$

ie., $x\,e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2}\,e^{\tan^{-1} y}\,dy + c$

By putting $\tan^{-1} y = t$, we obtain, (*As in the previous problem*)

$$x\,e^{\tan^{-1} y} = e^{\tan^{-1} y}(\tan^{-1} y - 1) + c,$$ which is the required solution.

Observe that N is a function of x only and M contains y. Thus the equation is linear in y.

>> We have $(1 + xy)\,dx = (1 + x^2)\,dy$ or $\frac{dy}{dx} = \frac{1 + xy}{1 + x^2}$

ie., $\frac{dy}{dx} - \frac{xy}{1+x^2} = \frac{1}{1+x^2}$

This equation is of the form $\frac{dy}{dx} + Py = Q$, where

$$P = \frac{-x}{1+x^2} \quad \text{and} \quad Q = \frac{1}{1+x^2}$$

$$\therefore \quad e^{\int P\,dx} = e^{-\int \frac{x\,dx}{1+x^2}} = e^{\frac{-1}{2}\log(1+x^2)} = \frac{1}{\sqrt{1+x^2}}$$

The solution is $y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$

ie., $\frac{y}{\sqrt{1+x^2}} = \int \frac{1}{(1+x^2)} \cdot \frac{1}{\sqrt{1+x^2}}\,dx + c$

ie., $\frac{y}{\sqrt{1+x^2}} = \int \frac{dx}{(1+x^2)^{3/2}} + c$

Put $x = \tan\theta \qquad \therefore \quad dx = \sec^2\theta\,d\theta$

Also $(1 + x^2)^{3/2} = (1 + \tan^2\theta)^{3/2} = \sec^3\theta$

$$\therefore \quad \frac{y}{\sqrt{1+x^2}} = \int \frac{\sec^2\theta}{\sec^3\theta}\,d\theta + c \quad \text{or} \quad \frac{y}{\sqrt{1+x^2}} = \int \cos\theta\,d\theta + c$$

ie., $$\frac{y}{\sqrt{1+x^2}} = \sin\theta + c \qquad \ldots(1)$$

But $\tan\theta = x \Rightarrow \cot\theta = 1/x$ and $1 + \cot^2\theta = 1 + (1/x^2)$

ie., $\operatorname{cosec}^2\theta = \dfrac{x^2+1}{x^2}$ or $\operatorname{cosec}\theta = \dfrac{\sqrt{x^2+1}}{x}$

$$\therefore \quad \sin\theta = \frac{x}{\sqrt{x^2+1}} \qquad \ldots (2)$$

Using (2) in (1) we get the general solution,

$$\frac{y}{\sqrt{1+x^2}} = \frac{x}{\sqrt{x^2+1}} + c \qquad \ldots (3)$$

But $x = 1$ and $y = 0$ and hence (3) will give us $c = -1/\sqrt{2}$

Hence (3) becomes,

$$\frac{y}{\sqrt{1+x^2}} = \frac{x}{\sqrt{x^2+1}} - \frac{1}{\sqrt{2}}$$

Thus $\boldsymbol{y = x - \sqrt{1+x^2/2}}$, is the required particular solution.

>> We have $dx = (e^{-y}\sec^2 y - x)\,dy$ or $\dfrac{dx}{dy} + x = e^{-y}\sec^2 y$

This is of the form $\dfrac{dx}{dy} + Px = Q$, where

$$P = 1 \text{ and } Q = e^{-y}\sec^2 y \quad \therefore \quad e^{\int P\,dy} = e^y$$

The solution $x\,e^{\int P\,dy} = \int Q\,e^{\int P\,dy}\,dy + c$ becomes

$$x\,e^y = \int e^{-y}\sec^2 y\, e^y\,dy + c$$

Thus $\boldsymbol{x\,e^y = \tan y + c}$, is the required solution.

Observe that M is a function of y and N contains x. (x dy is present). The equation is linear in x.

>> We have $y\,dx = (xy + 2 - 3x)\,dy$

ie., $\dfrac{dx}{dy} = \dfrac{xy+2-3x}{y}$ or $\dfrac{dx}{dy} = \dfrac{x(y-3)+2}{y}$

ie., $\dfrac{dx}{dy} - \dfrac{x(y-3)}{y} = \dfrac{2}{y}$

This equation is of the form $\dfrac{dx}{dy} + Px = Q$, where

$$P = -\frac{(y-3)}{y} = \left(\frac{3}{y}-1\right) \quad \text{and} \quad Q = \frac{2}{y}$$

Hence $\int P\,dy = \int\left(\frac{3}{y}-1\right)dy = 3\log y - y$

$$\therefore \quad e^{\int P\,dy} = e^{(3\log y - y)} = e^{3\log y} \cdot e^{-y} = y^3 e^{-y}$$

The solution is $x\,e^{\int P\,dy} = \int Q\,e^{\int P\,dy}\,dy + c$

ie., $\quad x\,y^3 e^{-y} = \int \frac{2}{y} \cdot y^3 e^{-y}\,dy + c$

ie., $\quad x\,y^3 e^{-y} = 2\int y^2 e^{-y}\,dy + c$

Applying Bernoulli's generalized rule of integration by parts we get,

$$x\,y^3 e^{-y} = 2\left\{y^2(-e^{-y}) - (2y)(e^{-y}) + 2(-e^{-y})\right\} + c$$

ie., $\quad x\,y^3 e^{-y} = -2e^{-y}(y^2 + 2y + 2) + c$

Thus $\quad x\,y^3 = -2(y^2 + 2y + 2) + c\,e^y$, is the required solution.

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68. [illegible]

>> We have $\dfrac{dx}{dy} = \dfrac{x + \tan y}{\sin 2y}$

ie., $\quad \dfrac{dx}{dy} - x\,\operatorname{cosec} 2y = \dfrac{\tan y}{\sin 2y}$

This equation is of the form $\dfrac{dx}{dy} + P\,x = Q$, where

$$P = -\operatorname{cosec} 2y \quad \text{and} \quad Q = \frac{\tan y}{\sin 2y} = \frac{\sin y}{\cos y \cdot 2\sin y\cos y} = \frac{1}{2}\sec^2 y$$

$$\therefore \quad e^{\int P\,dy} = e^{\int -\operatorname{cosec} 2y\,dy} = e^{-\frac{1}{2}\log(\tan y)} = \frac{1}{\sqrt{\tan y}}$$

where we have used $\int \operatorname{cosec} x\,dx = \log\tan(x/2)$

The solution is $x\,e^{\int P\,dy} = \int Q\,e^{\int P\,dy}\,dy + c$

ie., $\quad \dfrac{x}{\sqrt{\tan y}} = \displaystyle\int \frac{1}{2}\sec^2 y \cdot \frac{1}{\sqrt{\tan y}}\,dy + c$

Put $\quad \tan y = t \quad \therefore \quad \sec^2 y\,dy = dt$

Hence $\frac{x}{\sqrt{\tan y}} = \frac{1}{2}\int \frac{dt}{\sqrt{t}} + c$

ie., $\frac{x}{\sqrt{\tan y}} = \frac{1}{2}\frac{t^{1/2}}{(1/2)} + c$

ie., $\frac{x}{\sqrt{\tan y}} = \sqrt{\tan y} + c$

or $x = \tan y + c\sqrt{\tan y}$, is the required solution.

Observe that N is a function of x and M contains y. The equation is linear in y.

>> We have $dy - dx = \frac{2xy}{1+x^2}dx$

ie., $dy = \left(\frac{2xy}{1+x^2} + 1\right)dx$

ie., $\frac{dy}{dx} - \frac{2xy}{1+x^2} = 1$

This equation is of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{-2x}{1+x^2}$ and $Q = 1$.

Hence $\int P\,dx = \int \frac{-2x}{1+x^2}dx = -\log(1+x^2)$

$\therefore \quad e^{\int P\,dx} = e^{-\log(1+x^2)} = \frac{1}{1+x^2}$

The solution is $y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}dx + c$

ie., $\frac{y}{1+x^2} = \int 1 \cdot \frac{1}{1+x^2}dx + c$

Thus $\frac{y}{1+x^2} = \tan^{-1}x + c$, is the required solution.

>> We have $\frac{dy}{dx} = \frac{2\log x - y}{x\log x}$ or $\frac{dy}{dx} + \frac{y}{x\log x} = \frac{2}{x}$ is linear in y of the form

$\frac{dy}{dx} + Py = Q$, where $P = \frac{1}{x\log x}$ and $Q = \frac{2}{x}$

Hence $\int P\,dx = \int \frac{1/x}{\log x}dx = \log(\log x)$

$\therefore \quad e^{\int P\,dx} = e^{\log(\log x)} = \log x$

The solution is $\quad y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$

ie., $\quad y \log x = \int \frac{2}{x} \cdot \log x\,dx + c$

Put $\quad \log x = t \quad \therefore \quad \frac{1}{x}\,dx = dt$

Hence $\; y \log x = \int 2t\,dt + c \quad$ or $\quad y \log x = t^2 + c$

Thus $\quad \mathbf{y \log x = (\log x)^2 + c},$ is the required solution.

>> We have $\; \frac{dr}{d\theta} + (2\cot\theta)\,r = -\sin 2\theta$

Here $\quad P = P(\theta) = 2\cot\theta\,;\, Q = Q(\theta) = -\sin 2\theta = -2\sin\theta\cos\theta$

$$\int P\,d\theta = \int 2\cot\theta\,d\theta = 2\log(\sin\theta) = \log(\sin^2\theta)$$

$\therefore \quad e^{\int P\,d\theta} = e^{\log(\sin^2\theta)} = \sin^2\theta$

The solution is $\quad r\,e^{\int P\,d\theta} = \int Q\,e^{\int P d\theta}\,d\theta + c$

ie., $\quad r\sin^2\theta = \int (-2\sin\theta\cos\theta)\sin^2\theta\,d\theta + c$

ie., $\quad r\sin^2\theta = -2\int \sin^3\theta\cos\theta\,d\theta + c$

Put $\quad \sin\theta = t \quad \therefore \quad \cos\theta\,d\theta = dt$

Hence $\; r\sin^2\theta = -2\int t^3\,dt + c$

ie., $\quad r\sin^2\theta = -\frac{t^4}{2} + c \qquad$ or $\qquad r\sin^2\theta = \frac{-\sin^4\theta}{2.} + c$

Thus $\quad \mathbf{2\,r\sin^2\theta + \sin^4\theta = 2c},$ **is the required solution.**

Form (i) : $f'(y)\dfrac{dy}{dx}+Pf(y) = Q$, where P and Q are functions of x.

We put $f(y) = t \quad \therefore \quad f'(y)\dfrac{dy}{dx} = \dfrac{dt}{dx}$

The given equation becomes $\dfrac{dt}{dx}+Pt = Q$ which is a linear equation in t.

Similarly $f'(x)\dfrac{dx}{dy}+Pf(x) = Q$, where P and Q are functions of y can be reduced to the linear form by putting $f(x) = t$.

Form (ii) : $\dfrac{dy}{dx}+Py = Qy^n$, where P and Q are functions of x.

This equation is called as **Bernoulli's equation** in y.

We first divide the equation throughout by y^n to obtain

$$\frac{1}{y^n}\frac{dy}{dx}+Py^{1-n} = Q \qquad \ldots (1)$$

Put $\quad y^{1-n} = t \quad \therefore \quad (1-n)y^{-n}\dfrac{dy}{dx} = \dfrac{dt}{dx}$

or $\quad \dfrac{1}{y^n}\dfrac{dy}{dx} = \dfrac{1}{(1-n)}\dfrac{dt}{dx}$

Hence (1) becomes, $\dfrac{1}{(1-n)}\dfrac{dt}{dx}+Pt = Q$

or $\quad \dfrac{dt}{dx}+(1-n)P\cdot t = (1-n)Q$ which is a linear equation in t.

Similarly $\dfrac{dx}{dy}+Px = Qx^n$, where P and Q are function of y is called **Bernoulli's equation** in x. We first divide by x^n and later put $x^{1-n} = t$ to obtain a linear equation in t.

>> We have $e^y\dfrac{dy}{dx}+e^y = e^x \qquad \ldots (1)$

Put $\quad e^y = t \quad \therefore \quad e^y\dfrac{dy}{dx} = \dfrac{dt}{dx}$

Hence (1) becomes $\frac{dt}{dx} + t = e^x$

This equation is linear in t of the form $\frac{dt}{dx} + Pt = Q$, where

$P = 1$ and $Q = e^x$. $\quad \therefore \quad e^{\int P\,dx} = e^x$

The solution is $t\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$

ie., $\quad t\,e^x = \int e^x \cdot e^x\,dx + c$

ie., $\quad e^y e^x = \int e^{2x}\,dx + c$

Thus $\quad e^{x+y} = \frac{e^{2x}}{2} + c$, is the required solution.

Observing $\frac{dy}{dx}$ *in the equation, we shall get rid off* cos *y in the R.H.S of the given equation*

>> Dividing the given equation by cos y we have,

$$\sec y\ \tan y\ \frac{dy}{dx} + \sec y\ \tan x = \cos^2 x \qquad \ldots (1)$$

Now put $\sec y = t \quad \therefore \quad \sec y\ \tan y\ \frac{dy}{dx} = \frac{dt}{dx}$

Hence (1) becomes $\frac{dt}{dx} + t\tan x = \cos^2 x$

This equation is of the form $\frac{dt}{dx} + Pt = Q$, where we have,

$$P = \tan x \text{ and } Q = \cos^2 x.$$

$$\therefore \quad e^{\int P\,dx} = e^{\int \tan x\,dx} = e^{\log(\sec x)} = \sec x$$

The solution is $t\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$

ie., $\quad \sec y\ \sec x = \int \cos^2 x \cdot \sec x\,dx + c$

ie., $\quad \sec y \cdot \sec x = \int \cos x\,dx + c$

Thus $\quad \mathbf{\sec y\ \sec x = \sin x + c}$, is the required solution.

>> Dividing the given equation by $\cos^2 y$ we have,

$$\sec^2 y \frac{dy}{dx} + x \cdot \frac{2 \sin y \cos y}{\cos^2 y} = x^3$$

ie., $$\sec^2 y \frac{dy}{dx} + (2 \tan y)\, x = x^3 \qquad \ldots (1)$$

Put $\tan y = t \quad \therefore \; \sec^2 y \frac{dy}{dx} = \frac{dt}{dx}$

Hence (1) becomes $\frac{dt}{dx} + 2\,x\,t = x^3$.

This equation is of the form

$$\frac{dt}{dx} + P\,t = Q, \text{ where } P = 2x \text{ and } Q = x^3 \quad \therefore \quad e^{\int P\,dx} = e^{x^2}$$

The solution is $t\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$

ie., $$t\,e^{x^2} = \int x^3 e^{x^2}\,dx + c$$

Put $x^2 = u \quad \therefore \; 2x\,dx = du$ or $2x^3\,dx = x^2\,du = u\,du$

$\therefore$ $$t\,e^{x^2} = \int e^u \frac{u\,du}{2} \text{ or } t\,e^{x^2} = \frac{1}{2} \int u\,e^u\,du + c$$

ie., $t\,e^{x^2} = \frac{1}{2}(u\,e^u - e^u) + c$, on integration by parts.

Thus $\tan y\, e^{x^2} = \frac{e^{x^2}}{2}(x^2 - 1) + c$, is the required solution.

---

75. Solve [illegible]

>> The given equation be written in the form,

$$\cos y \frac{dy}{dx} + \sin y \cos x = \sin x \cos x \qquad \ldots (1)$$

Put $\sin y = t \quad \therefore \; \cos y \frac{dy}{dx} = \frac{dt}{dx}$

Hence (1) becomes,

$$\frac{dt}{dx} + t\cos x = \sin x\cos x$$

This is of the form $\frac{dt}{dx} + Pt = Q$ where we have,

$$P = \cos x,\ Q = \sin x\cos x$$

Solution is given by $t\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$

ie., $t\,e^{\sin x} = \int \sin x\,\cos x\,e^{\sin x}\,dx + c$

Put $\sin x = u \quad \therefore \quad \cos x\,dx = du$

$\therefore$ $t\,e^{\sin x} = \int u\,e^{u}\,du + c$

ie., $t\,e^{\sin x} = u\,e^{u} - e^{u} + c$

or $\mathbf{\sin y\,e^{\sin x} = e^{\sin x}(\sin x - 1) + c}$

>> Dividing the given equation by y^2 we have,

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{yx} = x \qquad \ldots (1)$$

Put $\frac{1}{y} = t \quad \therefore \quad \frac{-1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

Hence (1) becomes $\frac{-dt}{dx} + \frac{t}{x} = x$ or $\frac{dt}{dx} - \frac{t}{x} = -x$

This equation is a linear equation of the form $\frac{dt}{dx} + Pt = Q$, where

$$P = \frac{-1}{x} \text{ and } Q = -x.$$

$\therefore$ $e^{\int P\,dx} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$

The solution is $t\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$

ie., $t \cdot \frac{1}{x} = \int -x \cdot \frac{1}{x}\,dx + c$

Thus $\mathbf{\frac{1}{xy}} = -x + c$, is the required solution.

>> Dividing the given equation by y^3 we have,

$$\frac{1}{y^3}\frac{dy}{dx} - \frac{1}{2}\left(1+\frac{1}{x}\right)\frac{1}{y^2} = \frac{-3}{x} \quad \ldots (1)$$

Put $\frac{1}{y^2} = t \quad \therefore \quad \frac{-2}{y^3}\frac{dy}{dx} = \frac{dt}{dx}$ or $\frac{1}{y^3}\frac{dy}{dx} = \frac{-1}{2}\frac{dt}{dx}$

Hence (1) becomes $\frac{-1}{2}\frac{dt}{dx} - \frac{t}{2}\left(1+\frac{1}{x}\right) = \frac{-3}{x}$

or $\frac{dt}{dx} + \left(1+\frac{1}{x}\right)t = \frac{6}{x}$

This equation is of the form $\frac{dt}{dx} + Pt = Q$ where

$$P = 1+\frac{1}{x} \text{ and } Q = \frac{6}{x}$$

$$\therefore \quad e^{\int P\,dx} = e^{\int\left(1+\frac{1}{x}\right)dx} = e^{x+\log x} = e^x \cdot x$$

The solution is $t\,e^{\int P dx} = \int Q\,e^{\int P\,dx}\,dx + c$

ie., $t \cdot e^x x = \int \frac{6}{x} e^x x \; dx + c$

Thus $\frac{x\,e^x}{y^2} = 6\,e^x + c$, is the required solution.

>> Multiplying the given equation by y^2 we have,

$$y^2\frac{dy}{dx} - y^3 \tan x = \sin x \cos^2 x \quad \ldots (1)$$

Put $y^3 = t \quad \therefore \quad 3y^2\frac{dy}{dx} = \frac{dt}{dx}$ or $y^2\frac{dy}{dx} = \frac{1}{3}\frac{dt}{dx}$

Hence (1) becomes $\frac{1}{3}\frac{dt}{dx} - t\tan x = \sin x \cos^2 x$

or $\quad \dfrac{dt}{dx} - 3\tan x \cdot t = 3\sin x\cos^2 x$

This equation is of the form

$$\frac{dt}{dx} + P\,t = Q, \text{ where } P = -3\tan x \text{ and } Q = 3\sin x\cos^2 x$$

$$\therefore \quad e^{\int P\,dx} = e^{\int -3\tan x\,dx} = e^{-3\log(\sec x)} = (\sec x)^{-3} = \cos^3 x$$

The solution is $\; t\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$

ie., $\quad t\cos^3 x = \int 3\sin x\,\cos^2 x \cdot \cos^3 x\,dx + c$

ie., $\quad t\cos^3 x = 3\int \sin x\,\cos^5 x\,dx + c$

Put $\quad \cos x = u \quad \therefore \; -\sin x\,dx = du$

$\therefore \quad t\cos^3 x = -3\int u^5\,du + c$

ie., $\quad t\cos^3 x = -\dfrac{u^6}{2} + c$

Thus $\quad (y\cos x)^3 = \dfrac{-\cos^6 x}{2} + c$, is the required solution.

---

79. *Solve* : $r\sin\theta - \cos\theta\,\dfrac{dr}{d\theta} = r^2$

>> We have $\cos\theta\,\dfrac{dr}{d\theta} - r\sin\theta = -r^2$

Dividing by r^2 we get, $\dfrac{\cos\theta}{r^2}\,\dfrac{dr}{d\theta} - \dfrac{1}{r}\sin\theta = -1 \qquad \ldots(1)$

Put $\quad \dfrac{1}{r} = y$ and differentiate w.r.t θ

$\therefore \quad \dfrac{-1}{r^2}\,\dfrac{dr}{d\theta} = \dfrac{dy}{d\theta}$ and hence (1) becomes

$$-\cos\theta\,\frac{dy}{d\theta} - y\sin\theta = -1 \quad \text{or} \quad \frac{dy}{d\theta} + (\tan\theta)\,y = \sec\theta$$

This is of the form, $\dfrac{dy}{d\theta} + Py = Q$ where

$$P = P(\theta) = \tan\theta \text{ and } Q = Q(\theta) = \sec\theta$$

Here $e^{\int P\,d\theta} = e^{\int \tan\theta\,d\theta} = e^{\log(\sec\theta)} = \sec\theta$

Solution is given by $y\,e^{\int P\,d\theta} = \int Q\,e^{\int P\,d\theta}\,d\theta + c$

ie., $y\sec\theta = \int \sec^2\theta\,d\theta + c$

ie., $y\sec\theta = \tan\theta + c$

Thus $\dfrac{\sec\theta}{r} = \tan\theta + c$, is the required solution.

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50. [illegible]

Note that we cannot simplify the expression in respect of $\dfrac{dy}{dx}$

>> Consider $\dfrac{dx}{dy} = xy + x^2y^3$ or $\dfrac{dx}{dy} - xy = x^2y^3$. Dividing by x^2 we get,

$$\frac{1}{x^2}\frac{dx}{dy} - \frac{1}{x}y = y^3 \qquad \ldots(1)$$

Put $\dfrac{1}{x} = t \quad \therefore \quad \dfrac{-1}{x^2}\dfrac{dx}{dy} = \dfrac{dt}{dy}$

Hence (1) becomes $-\dfrac{dt}{dy} - ty = y^3$ or $\dfrac{dt}{dy} + ty = -y^3$

This equation is of the form $\dfrac{dt}{dy} + Pt = Q$, where

$$P = y \text{ and } Q = -y^3 \quad \therefore \quad e^{\int P\,dy} = e^{\int y\,dy} = e^{y^2/2}$$

The solution is $t\,e^{\int P\,dy} = \int Q\,e^{\int P\,dy}\,dy + c$

ie., $t\,e^{y^2/2} = -\int y^3 e^{y^2/2}\,dy + c$

Put $y^2/2 = u \quad \therefore \quad y\,dy = du$

Also $y^2 \cdot y\,dy = y^2\,du$ or $y^3\,dy = 2u\,du$

$\therefore \quad t\,e^{y^2/2} = -2\int u\,e^u\,du + c$

ie., $t\,e^{y^2/2} = -2(u\,e^u - e^u) + c$, on integration by parts.

Thus $\dfrac{e^{y^2/2}}{x} = 2\,e^{y^2/2}\left(1 - \dfrac{y^2}{2}\right) + c$, is the required solution.

81. Solve $(y \log x - 2)\, y\, dx = x\, dy$

>> The given equation can be written in the form

$$\frac{dy}{dx} = \frac{y\,(y \log x - 2)}{x}$$

ie., $\dfrac{dy}{dx} + \dfrac{2y}{x} = \dfrac{y^2 \log x}{x}$ Dividing by y^2 we get,

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{2}{xy} = \frac{\log x}{x} \qquad \ldots (1)$$

Put $\dfrac{1}{y} = t \quad \therefore \quad \dfrac{-1}{y^2}\dfrac{dy}{dx} = \dfrac{dt}{dx}$

Hence (1) becomes $-\dfrac{dt}{dx} + \dfrac{2t}{x} = \dfrac{\log x}{x}$

ie., $\dfrac{dt}{dx} - \dfrac{2t}{x} = -\dfrac{\log x}{x}$

This equation is of the form $\dfrac{dt}{dx} + P\,t = Q$, where

$$P = \frac{-2}{x} \text{ and } Q = -\frac{\log x}{x}$$

$$\therefore \quad e^{\int P\,dx} = e^{-\int \frac{2}{x} dx} = e^{-2\log x} = \frac{1}{x^2}$$

The solution is $t\, e^{\int P\,dx} = \int Q\, e^{\int P\,dx}\, dx + c$

ie., $\dfrac{t}{x^2} = -\displaystyle\int \log x \cdot \frac{1}{x^3}\, dx + c$

ie., $\dfrac{t}{x^2} = -\left\{\log x \cdot \dfrac{x^{-2}}{-2} - \displaystyle\int \frac{x^{-2}}{-2} \cdot \frac{1}{x}\, dx\right\} + c$

ie., $\dfrac{t}{x^2} = -\left\{-\dfrac{\log x}{2x^2} + \dfrac{1}{2}\,\dfrac{x^{-2}}{-2}\right\} + c$

ie., $\dfrac{t}{x^2} = \dfrac{1}{2x^2}\left(\log x + \dfrac{1}{2}\right) + c$

Thus $\dfrac{1}{x^2 y} = \dfrac{1}{2x^2}\left(\log x + \dfrac{1}{2}\right) + c$, is the required solution.

82. [illegible]

The given equation can be written in the form

$$\frac{dx}{dy} = \frac{x(x^3+2y)}{6y^2}$$

ie., $\dfrac{dx}{dy} - \dfrac{x}{3y} = \dfrac{x^4}{6y^2}$ Dividing by x^4 we have,

$$\frac{1}{x^4}\frac{dx}{dy} - \frac{1}{3x^3 y} = \frac{1}{6y^2} \qquad \ldots (1)$$

Put $\dfrac{1}{x^3} = t \quad \therefore \quad \dfrac{-3}{x^4}\dfrac{dx}{dy} = \dfrac{dt}{dy}$ or $\dfrac{1}{x^4}\dfrac{dx}{dy} = \dfrac{-1}{3}\dfrac{dt}{dy}$

Hence (1) becomes $\dfrac{-1}{3}\dfrac{dt}{dy} - \dfrac{t}{3y} = \dfrac{1}{6y^2}$ or $\dfrac{dt}{dy} + \dfrac{t}{y} = \dfrac{-1}{2y^2}$

This equation is of the form $\dfrac{dt}{dy} + Pt = Q$, where

$$P = \frac{1}{y} \text{ and } Q = \frac{-1}{2y^2} \quad \therefore \quad e^{\int P\,dy} = e^{\int \frac{1}{y}dy} = e^{\log y} = y$$

The solution is $t\,e^{\int P\,dy} = \int Q\,e^{\int P\,dy}\,dy + c$

ie., $ty = \int \dfrac{-1}{2y^2}\cdot y\;dy + c$

ie., $ty = -\dfrac{\log y}{2} + c$

Thus $\dfrac{y}{x^3} = -\dfrac{\log y}{2} + c$, is the required solution.

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83. Solve [illegible]

>> The given equation can be written in the form,

$$\frac{dy}{dx} = \frac{y(2xy+e^x)}{e^x}$$

ie., $\dfrac{dy}{dx} - y = \dfrac{2xy^2}{e^x}$ Dividing by y^2 we get,

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = \frac{2x}{e^x} \qquad \ldots (1)$$

Put $\dfrac{1}{y} = t \quad \therefore \quad \dfrac{-1}{y^2}\dfrac{dy}{dx} = \dfrac{dt}{dx}$

Hence (1) becomes $-\frac{dt}{dx} - t = \frac{2x}{e^x}$ or $\frac{dt}{dx} + t = -\frac{2x}{e^x}$

This equation is of the form $\frac{dt}{dx} + P\,t = Q$, where

$$P = 1 \text{ and } Q = \frac{-2x}{e^x} \quad \therefore \quad e^{\int P\,dx} = e^x$$

The solution is $t\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$

ie., $\quad t\,e^x = \int \frac{-2x}{e^x} \cdot e^x\,dx + c$

ie., $\quad t\,e^x = -x^2 + c$

ie., $\quad \frac{e^x}{y} + x^2 = c$, is the required solution.

84. Solve $x^3\frac{dy}{dx} - x^2 y = -y^4\cos x$

\>> We have $x^3\frac{dy}{dx} - x^2 y = -y^4\cos x$ Dividing by $x^3 y^4$ we get

$$\frac{1}{y^4}\frac{dy}{dx} - \frac{1}{x\,y^3} = -\frac{\cos x}{x^3} \qquad \ldots(1)$$

Put $\frac{1}{y^3} = t \quad \therefore \quad \frac{-3}{y^4}\frac{dy}{dx} = \frac{dt}{dx}$ or $\frac{1}{y^4}\frac{dy}{dx} = \frac{-1}{3}\frac{dt}{dx}$

Hence (1) becomes, $\frac{-1}{3}\frac{dt}{dx} - \frac{t}{x} = -\frac{\cos x}{x^3}$

ie., $\quad \frac{dt}{dx} + 3\frac{t}{x} = \frac{3\cos x}{x^3}$

This equation is a linear equation of the form $\frac{dt}{dx} + Pt = Q$, where

$$P = \frac{3}{x} \text{ and } Q = \frac{3\cos x}{x^3} \quad \therefore \quad e^{\int P\,dx} = e^{\int (3/x)\,dx} = e^{3\log x} = x^3$$

Solution is given by $t\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$

ie., $\quad t\,x^3 = \int \frac{3\cos x}{x^3} \cdot x^3\,dx + c$

ie., $\quad x^3/y^3 = 3\sin x + c$, is the required solution.

EXERCISES

Solve the following differential equations

1. $\dfrac{dy}{dx} - \dfrac{y}{x+1} = e^{3x}(x+1)$

2. $\dfrac{dy}{dx} + y\cot x = 4x \operatorname{cosec} x \; ; \; y(\pi/2) = 0$

3. $\dfrac{dy}{dx} = \dfrac{e^{-2\sqrt{x}}}{\sqrt{x}} - \dfrac{y}{\sqrt{x}}$

4. $x(x-1)\dfrac{dy}{dx} - y = \{x(x-1)\}^2$

5. $\sin 2x \dfrac{dy}{dx} - 2y = \tan x$

6. $(1+\sin x)\,dy + (x + y\cos x)\,dx = 0$ *(Solve by two methods)*

7. $xe^x \dfrac{dy}{dx} + e^x(1+x)\,y = 1$ *(Solve by two methods)*

8. $x\,dy + (y - x - xy\tan x)\,dx = 0$

9. $3y^2 \dfrac{dy}{dx} + 2xy^3 = 2xe^{-x^2}$

10. $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2 y^6$

11. $\dfrac{dy}{dx} + \dfrac{y\log y}{x} = y\left(\dfrac{\log y}{x}\right)^2$

12. $\dfrac{dy}{dx} + \dfrac{2y}{3} = \dfrac{x}{\sqrt{y}}$

13. $x^2 \cos y\,dy - (1 + 3x\sin y)\,dy = 0$

14. $(y^4 - 2xy)\,dx + 3x^2\,dy = 0 \; ; \; y(2) = 1$

15. $(x + \sqrt{xy})\,dy = y\,dx$ *(Solve by two methods)*

ANSWERS

1. $\dfrac{y}{x+1} = \dfrac{e^{3x}}{3} + c$

2. $y\sin x = 2x^2 - (\pi^2/2)$

3. $y e^{2\sqrt{x}} = 2\sqrt{x} + c$

4. $\dfrac{xy}{1-x} + \dfrac{x^3}{3} = c$

5. $y\cot x = \log\sqrt{\tan x} + c$

6. $y(1+\sin x) + \dfrac{x^2}{2} = c$

7. $xye^x = x + c$

8. $xy\cos x = \cos x + x\sin x + c$

9. $y^3 e^{x^2} = x^2 + c$

10. $\dfrac{1}{x^5 y^5} - \dfrac{5}{2x^2} = c$

11. $\dfrac{1}{x \log y} = \dfrac{1}{2x^2} + c$

12. $y^{3/2} e^x = \dfrac{3}{2}(x-1) e^x + c$

13. $4x \sin y + 1 = cx^4$

14. $x^2 = y^3 (x+2)$

15. $\sqrt{x} = \sqrt{y} (\log \sqrt{y} + c)$

* ***Refer articles 6.5 & 6.6 for Methods at a glance and Type recognition***

Orthogonal Trajectories

Introduction

Basically we know that, two curves intersect each other orthogonally if the tangents at the point of intersection are at right angles. Further we know from differential calculus that, for a cartesian curve $y = f(x)$, $m = \dfrac{dy}{dx}$ represents the slope of the tangent.

In order to show that two curves intersect orthogonally we simply obtain $\dfrac{dy}{dx}$ for the two curves say m_1, m_2 and establish $m_1 m_2 = -1$ being the condition for two lines to be perpendicular. In fact the orthogonality of two polar curves also has been discussed in chapter - 2 (*article* 2.2).

With the knowledge of differential equations, given a family of curves it is possible to determine another family of curves which intersects each member of the given family orthogonally and we discuss this concept in detail.

6.11 Orthogonal [illegible] polar family of curves

An equation of the form $f(x, y, c) = 0$ where c is a fixed constant represents a curve.

For example $x^2 + y^2 = 4$ is a circle, $y^2 = 4x$ is a parabola etc.

On the otherhand if c is an arbitrary constant (*c is a parameter*) the equation $f(x, y) = c$ represents a one parameter family of curves. For each value of c we get different curves of the same family.

For example $x^2 + y^2 = r^2$ (*r is arbitrary*) represents a family of concentric circles.

Definition : If two family of curves are such that every member of one family intersects every member of the other family at right angles then they are said to be *orthogonal trajectories* of each other.

Method of finding the orthogonal trajectories

Case - (i) Cartesian family $f(x, y, c) = 0$

We differentiate *w.r.t* x and eliminate the parameter c. The equation so obtained is called as the differential equation of the given family.

We know that if $\tan\psi = \frac{dy}{dx}$ is the slope of a given line then the slope of the line perpendicular to it is $\frac{-1}{\tan\psi} = -\frac{dx}{dy}$. Accordingly in the differential equation of the given family we shall replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ to arrive at a new differential equation. Solving this new differential equation we get the orthogonal trajectories of the given family of curves.

Self orthogonal family : If the differential equation of the given family remains unaltered after replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ then the given family of curves is said to be *self orthogonal.*

Case - (ii) : Polar family $f(r, \theta, c) = 0$

We know that $\tan\phi = r\frac{d\theta}{dr}$ for a polar curve where ϕ is the angle between the radius vector and the tangent. $\phi_2 - \phi_1 = 90^o$ is the condition for two polar curves to be orthogonal.

$$\therefore \quad \phi_2 = 90^o + \phi_1 \Rightarrow \tan\phi_2 = \tan(90^o + \phi_1)$$

ie., $$\tan\phi_2 = -\cot\phi_1 \quad \text{or} \quad \tan\phi_2 = \frac{-1}{\tan\phi_1}$$

But $\tan\phi_1 = r\frac{d\theta}{dr}$ for the given curve and $\tan\phi_2 = r\frac{d\theta}{dr}$ for the orthogonal curve at the same point.

$\therefore$ $r\frac{d\theta}{dr}$ for the curve to be replaced by $\dfrac{-1}{r\frac{d\theta}{dr}}$

ie., $-r^2\frac{d\theta}{dr}$ to be replaced by $\frac{dr}{d\theta}$ or vice - versa.

In otherwords, we have to differentiate $f(r, \theta, c) = 0$ *w.r.t* θ and eliminate c to obtain the D.E of the given family. We have to replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ to obtain the new D.E and solve the same to obtain the required orthogonal trajectories.

Working procedure for problems

Case - i : (Cartesian family)

- Given $f(x, y, c) = 0$, differentiate *w.r.t* x and eliminate c.
- Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ and solve the equation.

Case - ii : (Polar family)

- Given an equation in r and θ, we prefer to take logarithms first and then differentiate *w.r.t* θ.
- After ensuring that the given parameter is eliminated we replace $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$ and solve the equation.

A picturesque illustration of orthogonal trajectories (O. T)

Let us find the O.T of the family of circles $x^2 + y^2 = a^2$.

Differentiating *w.r.t* x we get, $2x + 2y\frac{dy}{dx} = 0$ or $x + y\frac{dy}{dx} = 0$.
This is the D.E of the given family.

Now, let us replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$\therefore \quad x + y\left(-\frac{dx}{dy}\right) = 0$ is the D.E of the orthogonal family.

ie., $\quad x\,dy - y\,dx = 0$. Dividing by $x\,y$ we get

$$\frac{dy}{y} - \frac{dx}{x} = 0$$

$$\Rightarrow \quad \int\frac{dy}{y} - \int\frac{dx}{x} = c$$

ie., $\quad \log y - \log x = c$ or $\log(y/x) = \log k$ *(say)*

ie., $\quad y/x = k$ or $y = kx$, is the required O.T.

Geometrically $y = kx$ for different values of k represent a family of straight lines passing through the origin. Let us draw a circle with centre origin and a straight line passing through the origin. (Figure-1)

Also, let us draw a family of circles and a family of straight lines of the same category. (Figure - 2)

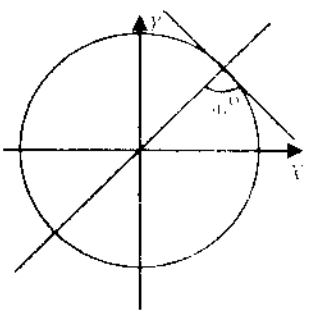


Figure - 1

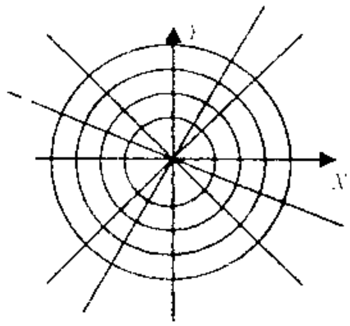


Figure - 2

It may be observed that in Figure - 1 the two curves, circle and the straight line intersect on the circumference of the circle and a tangent is drawn at the point of intersection. The angle of intersection can easily be seen as 90^0. In Figure - 2 we observe that any straight line passing through the origin intersects every member of the family of circles at right angles and vice-versa.

Hence, theoretically and geometrically we can say that the family of circles $(x^2+y^2=a^2)$ and the family of straight lines passing through the origin $(y=kx)$ are orthogonal trajectories of each other.

WORKED PROBLEMS

85. *Find the O.T of the family of parabolas* $y^2 = 4ax$.

>> Consider $\dfrac{y^2}{x} = 4a$ ... (1)

(If the parameter is on one side of the equation exclusively, then the same gets eliminated once we differentiate)

Now differentiating (1) *w.r.t* x we have

$$\frac{x\cdot 2y\,\dfrac{dy}{dx} - y^2\cdot 1}{x^2} = 0 \quad \text{or} \quad 2xy\,\frac{dy}{dx} - y^2 = 0$$

ie., $2x\dfrac{dy}{dx} - y = 0$, is the D.E of the given family.

Replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ we have,

$$2x\left(-\frac{dx}{dy}\right) - y = 0 \quad \text{or} \quad 2x\,dx + y\,dy = 0$$

$$\Rightarrow \quad \int 2x\,dx + \int y\,dy = c$$

ie., $x^2 + \frac{y^2}{2} = c$ or $2x^2 + y^2 = 2c = k$ (say)

Thus $2x^2 + y^2 = k$, is the required O.T.

---

86. *Find the O.T of the family of astroids* $x^{2/3} + y^{2/3} = a^{2/3}$

>> Consider $x^{2/3} + y^{2/3} = a^{2/3}$

Differentiating *w.r.t x,* we have

$$\frac{2}{3} \cdot x^{-1/3} + \frac{2}{3} \cdot y^{-1/3} \frac{dy}{dx} = 0$$

ie., $x^{-1/3} + y^{-1/3} \frac{dy}{dx} = 0$, is the D.E of the given family.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ we have,

$$x^{-1/3} + y^{-1/3}\left(-\frac{dx}{dy}\right) = 0 \quad ie., \; x^{-1/3}\, dy = y^{-1/3}\, dx$$

ie., $y^{1/3}\, dy = x^{1/3}\, dx$ by separating the variables.

$\Rightarrow$ $\int y^{1/3}\, dy - \int x^{1/3}\, dx = c$

ie., $\frac{y^{4/3}}{(4/3)} - \frac{x^{4/3}}{(4/3)} = c$ or $x^{4/3} - y^{4/3} = -\frac{4c}{3} = k$ (say)

Thus $x^{4/3} - y^{4/3} = k$ is the required O.T.

---

87. *Find the O.T of the family* $y^2 = c\,x^3$.

>> We have $\frac{y^2}{x^3} = c$

Differentiating *w.r.t x* we have,

$$\frac{x^3 \cdot 2y \frac{dy}{dx} - y^2 \cdot 3x^2}{x^6} = 0 \qquad \text{or} \qquad 2x^3 y \frac{dy}{dx} = 3x^2 y^2$$

ie., $2x \frac{dy}{dx} = 3y$ Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ we have,

$$2x\left(-\frac{dx}{dy}\right) = 3y$$

ie., $2x\, dx + 3y\, dy = 0$

$\Rightarrow \quad \int 2x\,dx + \int 3y\,dy = c$

ie., $\quad x^2 + \dfrac{3y^2}{2} = c$ or $2x^2 + 3y^2 = 2c = k$ *(say)*

Thus $2x^2 + 3y^2 = k$, **is the required O.T.**

---

88. *Show that the family of curves $x^3 - 3xy^2 = c_1$ and $y^3 - 3x^2y = c_2$ are O.T of each other.*

>> Let us consider $x^3 - 3xy^2 = c_1$ and differentiate *w.r.t x*.

$\therefore \quad 3x^2 - 3\left(x \cdot 2y\dfrac{dy}{dx} + y^2\right) = 0$

ie., $\quad x^2 - y^2 = 2xy\dfrac{dy}{dx}$, is the D.E of the given family.

Replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ we have,

$$x^2 - y^2 = 2xy\left(-\frac{dx}{dy}\right) \quad \text{or} \quad 2xy\,dx + (x^2 - y^2)\,dy = 0$$

(This is a homogeneous equation. But it is also exact)

Let $\quad M = 2xy$ and $N = x^2 - y^2$

$\therefore \quad \dfrac{\partial M}{\partial y} = 2x$ and $\dfrac{\partial N}{\partial x} = 2x$. Hence the equation is exact.

The solution is given by $\int M\,dx + \int N(y)\,dy = c$

ie., $\quad \int 2xy\,dx + \int -y^2\,dy = c$

ie., $\quad x^2y - \dfrac{y^3}{3} = c$ or $3x^2y - y^3 = 3c$

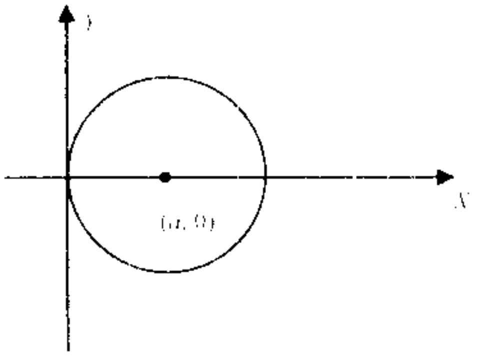
$\therefore \quad y^3 - 3x^2y = c_2$ *(say)* is the required O.T where $c_2 = -3c$

Thus $x^3 - 3xy^2 = c_1$ and $y^3 - 3x^2y = c_2$ are orthogonal trajectories of each other.

---

89. *Show that the orthogonal trajectories of a family of circles passing through the origin having their centres on the x - axis is a family of circles passing through the origin having their centres on the y - axis.*

>> We shall draw a circle of the given family so as to write its equation.



Centre = $(a, 0)$ and radius is 'a' since the circle passes through the origin.

$\therefore$ Equation of the given family of circles is

$$(x-a)^2+(y-0)^2 = a^2$$

ie., $\quad x^2-2ax+y^2 = 0 \qquad \ldots(1)$

Differentiating *w.r.t* x we have,

$$2x-2a+2yy_1 = 0 \quad \text{where} \quad y_1 = \frac{dy}{dx}$$

$\therefore \quad a = x+yy_1$ and substituting this value of 'a' in (1) we get,

$$x^2-2(x+yy_1)x+y^2 = 0$$

ie., $y^2-x^2 = 2xyy_1$ is the D.E of the given family.

Now let us replace $y_1 = \dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$

$$\therefore \quad y^2-x^2 = 2xy\left(-\frac{dx}{dy}\right)$$

or $\quad \dfrac{dx}{dy} = \dfrac{x^2-y^2}{2xy}$ which is a homogeneous equation.

Put $\quad x = vy \quad \therefore \quad \dfrac{dx}{dy} = v+y\dfrac{dv}{dy}$

Hence $v+y\dfrac{dv}{dy} = y^2\dfrac{(v^2-1)}{2vy^2}$

ie., $y\dfrac{dv}{dy} = \dfrac{v^2-1}{2v} - v$ or $y\dfrac{dv}{dy} = -\dfrac{(1+v^2)}{2v}$

$\therefore$ $\dfrac{2v\,dv}{1+v^2} = -\dfrac{dy}{y}$ by separating the variables.

$\Rightarrow$ $\displaystyle\int \frac{2v}{1+v^2}\,dv + \int \frac{dy}{y} = c$

ie., $\log(1+v^2) + \log y = c$

ie., $\log\left[(1+v^2)\,y\right] = \log k$ (*say*) where $v = x/y$

$\therefore$ $\left(1+\dfrac{x^2}{y^2}\right)y = k$ or $x^2+y^2 = ky$

ie., $x^2+y^2-ky = 0$ and $k = 2b$ for convenience.

$\therefore$ $x^2+y^2-2by = 0$ or $x^2+(y-b)^2-b^2 = 0$

ie., $x^2+(y-b)^2 = b^2$

This is the equation of the family of circles passing through the origin having their centres on the y - axis.

Thus we have proved the required result.

---

90. *Find the orthogonal trajectories of the family of curves* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2+\lambda} = 1$, *where* λ *is the parameter.*

>> We have $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2+\lambda} = 1$ ... (1)

Differentiating $w.r.t\ x$ we have,

$$\frac{2x}{a^2} + \frac{2yy_1}{b^2+\lambda} = 0, \text{ where } y_1 = \frac{dy}{dx}$$

ie., $\dfrac{x}{a^2} = \dfrac{-yy_1}{b^2+\lambda}$ ... (2)

Also from (1) $\dfrac{x^2}{a^2} - 1 = \dfrac{-y^2}{b^2+\lambda}$ or $\dfrac{x^2-a^2}{a^2} = \dfrac{-y^2}{b^2+\lambda}$ ... (3)

Now, dividing (2) by (3) we get,

$$\frac{x}{x^2-a^2} = \frac{yy_1}{y^2} \text{ or } \frac{x}{x^2-a^2} = \frac{y_1}{y}$$

Now let us replace $y_1 = \frac{dy}{dx}$ by $-\frac{dx}{dy}$

$\therefore \quad \frac{x}{x^2 - a^2} = \frac{1}{y}\left(-\frac{dx}{dy}\right)$ or $y\,dy = -\frac{(x^2 - a^2)}{x}\,dx$ by separating the variables.

$\Rightarrow \quad \int y\,dy = -\int x\,dx + a^2 \int \frac{dx}{x} + c$

ie., $\quad \frac{y^2}{2} = \frac{-x^2}{2} + a^2 \log x + c$

Thus $x^2 + y^2 - 2a^2 \log x - b = 0$ where $b = 2c$, is the required orthogonal trajectories.

---

91. *Find the orthogonal trajectories of the family of coaxal circles* $x^2 + y^2 + 2\lambda x + c = 0$, λ *being the parameter.*

OR

Show that the orthogonal trajectories of a family of coaxal circles $x^2 + y^2 + 2\lambda x + c = 0$, λ *being the parameter is also a system of coaxial circles*

>> Consider $x^2 + y^2 + 2\lambda x + c = 0$ ... (1)

Differentiating *w.r.t* x we have,

$$2x + 2y\,y_1 + 2\lambda = 0, \text{ where } y_1 = \frac{dy}{dx}$$

$\therefore \quad \lambda = -(x + y\,y_1)$ and substituting this value in (1) we get,

$$(x^2 + y^2) - 2(x + yy_1)x + c = 0$$

ie., $\quad y^2 - x^2 - 2xy\,y_1 + c = 0$ or $-2xy\,y_1 = x^2 - y^2 - c$

$$\therefore \quad -y_1 = \frac{x^2 - y^2 - c}{2xy}$$

Now replacing $y_1 = \frac{dy}{dx}$ by $-\frac{dx}{dy}$ we have

$$\frac{dx}{dy} = \frac{x^2 - y^2 - c}{2xy} \quad \text{or} \quad \frac{dx}{dy} = \frac{x}{2y} - \frac{(y^2 + c)}{2xy}$$

ie., $\quad 2x\frac{dx}{dy} - \frac{x^2}{y} = -\left(y + \frac{c}{y}\right)$ ... (2)

Now put $x^2 = t \quad \therefore \quad 2x\dfrac{dx}{dy} = \dfrac{dt}{dy}$

Hence (2) becomes $\dfrac{dt}{dy} - \dfrac{t}{y} = -\left(y + \dfrac{c}{y}\right)$

This is a linear equation of the form

$$\frac{dt}{dy} + Pt = Q, \text{ where } P = \frac{-1}{y} \text{ and } Q = -\left(y + \frac{c}{y}\right)$$

Hence $e^{\int P\,dy} = e^{\int \frac{-1}{y}dy} = e^{-\log y} = \dfrac{1}{y}$

The solution is $t\,e^{\int P\,dy} = \int Q\,e^{\int P\,dy}\,dy + c_1$

ie., $$\frac{t}{y} = \int -\left(y + \frac{c}{y}\right)\frac{1}{y}\,dy + c_1$$

ie., $$\frac{t}{y} = -\int dy - c\int \frac{1}{y^2}\,dy + c_1$$

ie., $$\frac{x^2}{y} = -y + \frac{c}{y} + c_1 \quad \text{or} \quad x^2 = -y^2 + c + c_1 y$$

$$\therefore \quad x^2 + y^2 - c_1 y - c = 0 \;;\; \text{Let } -c_1 = 2\mu \text{ and } -c = c_2$$

Then $\mathbf{x^2 + y^2 + 2\mu y + c_2 = 0}$ (μ *is the parameter*) is the required orthogonal trajectories which is also a coaxial system of circles.

---

92. *If $u(x, y)$ and $v(x, y)$ are two functions satisfying the conditions :*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \textit{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

verify that the family of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ where c_1 and c_2 are arbitrary constants are orthogonal trajectories of each other.

>> It is enough if we show that the product of the slope of the tangents of the two family of curves at the point of intersection is equal to -1.

Consider $u(x, y) = c_1$ and differentiate *w.r.t* x treating y as a function of x. This will give us

$$u_x + u_y\frac{dy}{dx} = 0, \text{ where } u_x = \frac{\partial u}{\partial x} \text{ and } u_y = \frac{\partial u}{\partial y}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{u_x}{u_y} = m_1 \text{ (say)}$$

Similarly for the family $v(x, y) = c_2$ we can obtain

$$\frac{dy}{dx} = -\frac{v_x}{v_y} = m_2 \quad (say)$$

Now consider the product of the slopes of the tangents which being $m_1 \cdot m_2$

$$\therefore \qquad m_1 \cdot m_2 = -\frac{u_x}{u_y} \cdot -\frac{v_x}{v_y} = \frac{u_x \cdot v_x}{u_y \cdot v_y}$$

But, we have by data $u_x = v_y$ and $v_x = -u_y$

Hence $m_1 \cdot m_2 = \dfrac{v_y \cdot -u_y}{u_y \cdot v_y} = -1$

This proves the required result.

---

93. *Show that the family of parabolas $y^2 = 4a(x+a)$ is self orthogonal.*

>> Consider $y^2 = 4a(x+a)$ ... (1)

Differentiating *w.r.t.* x, we have

$$2y\frac{dy}{dx} = 4a \quad \therefore \quad a = \frac{y\,y_1}{2} \quad \text{where} \quad y_1 = \frac{dy}{dx}$$

Substituting this value of 'a' in (1) we have,

$$y^2 = 2\,y\,y_1\left(x + \frac{y\,y_1}{2}\right) \quad \text{or} \quad y = 2\,x\,y_1 + y\,y_1^2$$

Thus we have, $y = 2\,x\,y_1 + y\,y_1^2$ ... (2)

This is the D.E of the given family.

Now replacing y_1 by $-1/y_1$, (2) becomes

$$y = 2\,x\left(\frac{-1}{y_1}\right) + y\left(\frac{-1}{y_1}\right)^2 \quad \text{or} \quad y = \frac{-2x}{y_1} + \frac{y}{y_1^2}$$

$$\therefore \qquad y\,y_1^2 + 2\,x\,y_1 = y \qquad \text{... (3)}$$

(3) the D.E. of the orthogonal family which is same as (2) being the D.E. of the given family.

Thus the family of parabolas $y^2 = 4\,a(x+a)$ is self orthogonal.

---

94. *Find the orthogonal trajectories of the family of curves* $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$, *where* λ *is a parameter.*

OR

Show that the family of conics $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$, *where* λ *is a parameter is self orthogonal.*

>> Consider $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$ ... (1)

Differentiating $w.r.t\ x$ we have,

$$\frac{2x}{a^2+\lambda}+\frac{2y\,y_1}{b^2+\lambda}=0 \quad \text{or} \quad \frac{x}{a^2+\lambda}=\frac{-y y_1}{b^2+\lambda} \qquad \ldots (2)$$

We have to eliminate λ to obtain the D.E of the given family.

We have a property in ratio and proportion that if $\frac{a}{b}=\frac{c}{d}$ then it is also equal to $\frac{a-c}{b-d}$. That is $\frac{a}{b}=\frac{c}{d}$ Also $\frac{a-c}{b-d}$

Hence (2) becomes,

$$\frac{x}{a^2+\lambda}=\frac{-y\,y_1}{b^2+\lambda}=\frac{x+y\,y_1}{a^2-b^2}$$

$$\Rightarrow \quad \frac{x}{a^2+\lambda}=\frac{x+yy_1}{a^2-b^2} \quad \text{and} \quad \frac{-yy_1}{b^2+\lambda}=\frac{x+yy_1}{a^2-b^2}$$

ie.,

$$\frac{x}{a^2+\lambda}=\frac{x+y\,y_1}{a^2-b^2} \qquad \ldots (3)$$

and

$$\frac{y}{b^2+\lambda}=\frac{x+y\,y_1}{-y_1\,(a^2-b^2)} \qquad \ldots (4)$$

Now $(3)\times x+(4)\times y$ will give us

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=\frac{x\,(x+y\,y_1)}{(a^2-b^2)}+\frac{y\,(x+y\,y_1)}{-y_1\,(a^2-b^2)}$$

Using (1) in the L.H.S we have,

$$1 = \frac{x + y\,y_1}{a^2 - b^2}\left(x - \frac{y}{y_1}\right) \quad \text{or} \quad (x + y\,y_1)\left(x - \frac{y}{y_1}\right) = a^2 - b^2 \qquad \ldots (5)$$

This is the D.E of the given family.

Now replacing y_1 by $-1/y_1$ in (5) we get the D.E of the orthogonal family.

ie., $$\left(x - \frac{y}{y_1}\right)(x + y\,y_1) = (a^2 - b^2) \qquad \ldots (6)$$

It may be observed that (5) and (6) are identically equal. In otherwords the D.E of the given family and the orthogonal family are the same.

Hence the given family of conics is self orthogonal.

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95. *Given $y = k e^{-2x} + 3x$, find a member of its orthagonal trajectories passing through the point $(0, 3)$.*

>> $$y = k e^{-2x} + 3x \qquad \ldots (1)$$

$$\frac{dy}{dx} = -2ke^{-2x} + 3$$

ie., $$\frac{dy}{dx} = -2(y - 3x) + 3, \text{ by using (1). [} k \text{ is eliminated]}$$

This is the *d.e* of the given family and now replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ we have,

$$-\frac{dx}{dy} = -2(y - 3x) + 3 \quad \text{or} \quad \frac{dx}{dy} = 2y - (6x + 3)$$

is the *d.e* of the orthogonal family which has to be solved.

We have $\frac{dx}{dy} + 6x = (2y - 3)$ which is of the form

$$\frac{dx}{dy} + Px = Q; \text{ where } P = 6 \text{ and } Q = (2y - 3)$$

Solution : $x\,e^{\int P\,dy} = c + \int Q\,e^{\int P\,dy}\,dy$

ie., $$x\,e^{6y} = c + \int (2y - 3)\,e^{6y}\,dy$$

ie., $$x\,e^{6y} = c + \left\{(2y - 3)\frac{e^{6y}}{6} - (2)\frac{e^{6y}}{36}\right\}, \text{ by the product rule.}$$

ie., $$x\,e^{6y} = c + \left\{ (2y-3)\,\frac{e^{6y}}{6} - \frac{e^{6y}}{18} \right\} \qquad \ldots (2)$$

This is the O.T of the given family. Further to find a member of this family passing through the point (0 , 3) we have from (2),

$$0 = c + \left\{ 3\,\frac{e^{18}}{6} - \frac{e^{18}}{18} \right\}$$

ie., $$-c = e^{18}\left(\frac{1}{2} - \frac{1}{18}\right) \text{ or } -c = \frac{8}{18}\,e^{18} \quad \therefore \quad c = -\frac{4}{9}\,e^{18}$$

Using this value of c in (2) we have,

$$x\,e^{6y} = -\frac{4}{9}\,e^{18} + \left\{ (2y-3)\,\frac{e^{6y}}{6} - \frac{e^{6y}}{18} \right\}$$

or $$18\,x\,e^{6y} = -8\,e^{18} + e^{6y}\,(6y-10)$$

Thus $9x\,e^{6y} = -4\,e^{18} + (3y-5)\,e^{6y}$ is the member of the orthogonal family passing through the point (0 , 3).

---

96. *Find the O.T of the family* $r = a\,(1+\sin\theta)$

>> We have $r = a\,(1+\sin\theta)$

$\Rightarrow$ $$\log r = \log a + \log(1+\sin\theta)$$

Differentiating *w.r.t* θ we have,

$$\frac{1}{r}\,\frac{dr}{d\theta} = 0 + \frac{\cos\theta}{1+\sin\theta}$$

Replacing $\dfrac{dr}{d\theta}$ by $-r^2\,\dfrac{d\theta}{dr}$ we get,

$$\frac{1}{r}\left(-r^2\,\frac{d\theta}{dr}\right) = \frac{\cos\theta}{1+\sin\theta}$$

ie., $$-r\,\frac{d\theta}{dr} = \frac{\cos\theta}{1+\sin\theta} \quad \text{or} \quad \frac{1+\sin\theta}{\cos\theta}\,d\theta = -\frac{dr}{r}$$ by separating the variables.

Hence $$\int \frac{dr}{r} + \int \frac{1+\sin\theta}{\cos\theta}\,d\theta = c$$

ie., $$\log r + \int \sec\theta\,d\theta + \int \tan\theta\,d\theta = c$$

ie., $$\log r + \log(\sec\theta + \tan\theta) + \log(\sec\theta) = c$$

ie., $$\log\left[\,r\,(\sec\theta + \tan\theta)\,\sec\theta\,\right] = \log b \ (say)$$

$\Rightarrow \quad r\left(\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}\right)\frac{1}{\cos\theta} = b$

ie., $\quad \frac{r(1+\sin\theta)}{\cos^2\theta} = b \quad$ or $\quad \frac{r(1+\sin\theta)}{(1-\sin^2\theta)} = b$

Thus $\mathbf{r = b(1-\sin\theta)}$ **is the required O.T.**

---

97. *Find the O.T of the family* $2a/r = 1-\cos\theta$

\>> We have $2a/r = 1-\cos\theta$

$\Rightarrow \quad \log 2a - \log r = \log(1-\cos\theta)$

Differentiating *w.r.t* θ we have,

$$\frac{-1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta}$$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ and simplifying R.H.S we have

$$\frac{-1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)}$$

ie., $\quad r\frac{d\theta}{dr} = \cot(\theta/2) \quad$ or $\quad \tan(\theta/2)\,d\theta = \frac{dr}{r}$

$\therefore \quad \int\frac{dr}{r} - \int\tan(\theta/2)\,d\theta = c$

ie., $\log r - \frac{\log\sec(\theta/2)}{(1/2)} = c$

ie., $\quad \log\left[r/\sec^2(\theta/2)\right] = \log b$ (*say*)

$\Rightarrow \quad r/\sec^2(\theta/2) = b \quad$ or $\quad r\cos^2(\theta/2) = b$

This is the required equation of the O.T which can be put in the following form.

$r\cdot\frac{1}{2}(1+\cos\theta) = b$ or $r(1+\cos\theta) = 2b$

Thus $\mathbf{2b/r = 1+\cos\theta}$ **is the required O.T.**

---

98. *Find the O.T of the family* $r^n \cos n\theta = a^n$

>> We have $r^n \cos n\theta = a^n$

$\Rightarrow \quad n \log r + \log(\cos n\theta) = n \log a$

Differentiating *w.r.t* θ we have,

$$\frac{n}{r}\frac{dr}{d\theta} + \left(\frac{-n\sin n\theta}{\cos n\theta}\right) = 0 \quad \text{or} \quad \frac{1}{r}\frac{dr}{d\theta} = \tan n\theta$$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ we have,

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \tan n\theta \quad \text{or} \quad -r\frac{d\theta}{dr} = \tan n\theta$$

$\therefore \quad \frac{d\theta}{\tan n\theta} = \frac{dr}{-r}$ by separating the variables.

$\Rightarrow \quad \int\frac{dr}{r} + \int\cot n\theta\, d\theta = c$

ie., $\quad \log r + \frac{1}{n}\log(\sin n\theta) = c \quad$ or $\quad n\log r + \log(\sin n\theta) = nc$

ie., $\quad \log(r^n \sin n\theta) = \log b$ *(say)*

Thus $r^n \sin n\theta = b$, **is the required O.T.**

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99. *Find the orthogonal trajectories of the family of curves* $r^n = a^n \cos n\theta$

>> $r^n = a^n \cos n\theta$

$\Rightarrow \quad n\log r = n\log a + \log(\cos n\theta)$

Differentiating w.r.t θ we get,

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{-n\sin n\theta}{\cos n\theta} \quad \text{or} \quad \frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ we get

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\tan n\theta$$

ie., $\quad r\frac{d\theta}{dr} = \tan n\theta$

$\therefore \quad \frac{d\theta}{\tan n\theta} = \frac{dr}{r} \Rightarrow \int\frac{dr}{r} - \int\cot n\theta\, d\theta = c$

ie., $\log r - \frac{1}{n}\log(\sin n\theta) = c$ or $n\log r - \log(\sin n\theta) = nc$

ie., $\log\left[\frac{r^n}{\sin n\theta}\right] = \log k\,(say) \Rightarrow r^n = k\sin n\theta$

Thus $r^n = k\sin n\theta$, is the required orthogonal trajectory.

---

100. *Show that the orthogonal trajectories of the family of cardioides $r = a\cos^2(\theta/2)$ is another family of cardioides $r = b\sin^2(\theta/2)$*

\>> We have $r = a\cos^2(\theta/2)$

$\Rightarrow$ $\log r = \log a + 2\log\cos(\theta/2)$

Differentiating *w.r.t* θ we have,

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + 2\cdot\frac{1}{\cos(\theta/2)}\cdot -\sin(\theta/2)\cdot\frac{1}{2}$$

ie., $$\frac{1}{r}\frac{dr}{d\theta} = -\tan(\theta/2)$$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ we have,

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\tan(\theta/2)$$

ie., $-r\frac{d\theta}{dr} = -\tan(\theta/2)$ or $\cot(\theta/2)\,d\theta = \frac{dr}{r}$ by separating the variables.

$\therefore$ $\int\frac{dr}{r} - \int\cot(\theta/2)\,d\theta = c$

ie., $\log r - 2\log\sin(\theta/2) = c$

ie., $\log\left(\frac{r}{\sin^2(\theta/2)}\right) = \log b\ (say)$

Thus $r = b\sin^2(\theta/2)$ **is the required O.T.**.

---

101. *Find the orthogonal trajectories of the family of curves $r = 4a\sec\theta\tan\theta$.*

\>> We have $r = 4a\sec\theta\tan\theta$

$\Rightarrow$ $\log r = \log(4a) + \log(\sec\theta) + \log(\tan\theta)$

Differentiating *w.r.t* θ we have,

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\sec\theta\tan\theta}{\sec\theta} + \frac{\sec^2\theta}{\tan\theta}$$

ie., $$\frac{1}{r}\frac{dr}{d\theta} = \frac{\tan^2\theta + \sec^2\theta}{\tan\theta}$$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ we have,

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \frac{\tan^2\theta + \sec^2\theta}{\tan\theta}$$

or $$\frac{\tan\theta}{\tan^2\theta + \sec^2\theta}\,d\theta = \frac{dr}{-r}$$ by separating the variables

$\Rightarrow$ $$\int\frac{dr}{r} + \int\frac{\tan\theta}{\tan^2\theta + \sec^2\theta}\,d\theta = c$$

ie., $$\log r + \int\frac{\sin\theta/\cos\theta}{\sin^2\theta/\cos^2\theta + 1/\cos^2\theta}\,d\theta = c$$

ie., $$\log r + \int\frac{\sin\theta\cos\theta}{\sin^2\theta + 1}\,d\theta = c$$

ie., $$\log r + \frac{1}{2}\log(1 + \sin^2\theta) = c$$

ie., $$\log\left[r\sqrt{1+\sin^2\theta}\right] = \log b \quad (say) \quad \Rightarrow \quad r\sqrt{1+\sin^2\theta} = b$$

Thus $\mathbf{r^2(1+\sin^2\theta) = b^2}$, is the required O.T.

---

102. *Find the orthogonal trajectories of the family of Lemniscates* $r^2 = a^2\cos 2\theta$

>> We have, $r^2 = a^2\cos 2\theta$

$\Rightarrow$ $2\log r = 2\log a + \log(\cos 2\theta)$

Differentiating *w.r.t* θ we get,

$$\frac{2}{r}\frac{dr}{d\theta} = \frac{-2\sin 2\theta}{\cos 2\theta} \quad or \quad \frac{1}{r}\frac{dr}{d\theta} = -\tan 2\theta$$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ we get,

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\tan 2\theta \quad \text{or} \quad r\frac{d\theta}{dr} = \tan 2\theta$$

$\therefore$ $$\frac{d\theta}{\tan 2\theta} = \frac{dr}{r},$$ by separating the variables.

$\Rightarrow \quad \int \frac{dr}{r} - \int \cot 2\theta \, d\theta = c$

ie., $\quad \log r - \frac{1}{2}\log(\sin 2\theta) = c$

ie., $\quad \log\left[\frac{r}{\sqrt{\sin 2\theta}}\right] = \log b \text{ (say)} \Rightarrow \quad r = b\sqrt{\sin 2\theta}$

Thus $\quad \mathbf{r^2 = b^2 \sin 2\theta}$, is the required O.T

103. *Find the orthogonal trajectories of the family of curves* $\left(r + \frac{k^2}{r}\right)\cos\theta = a$, *a being the parameter*

>> We have $\left(r + \frac{k^2}{r}\right)\cos\theta = a$

$\Rightarrow \quad \log\left(r + \frac{k^2}{r}\right) + \log\cos\theta = \log a$

Differentiating *w.r.t* θ we have,

$$\frac{1}{\left(r + \frac{k^2}{r}\right)}\left(1 - \frac{k^2}{r^2}\right)\frac{dr}{d\theta} + \left(-\frac{\sin\theta}{\cos\theta}\right) = 0$$

ie., $\quad \frac{r}{r^2 + k^2} \cdot \frac{r^2 - k^2}{r^2}\frac{dr}{d\theta} = \tan\theta$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ we have,

$$\frac{r}{r^2 + k^2} \cdot \frac{r^2 - k^2}{r^2}\left(-r^2\frac{d\theta}{dr}\right) = \tan\theta$$

ie., $\quad \frac{r(r^2 - k^2)}{r^2 + k^2}\left(-\frac{d\theta}{dr}\right) = \tan\theta$

or $\quad -\frac{d\theta}{\tan\theta} = \frac{r^2 + k^2}{r(r^2 - k^2)}\,dr$ by separating the variables

$\Rightarrow \quad \int \frac{(r^2 + k^2)}{r(r^2 - k^2)}\,dr + \int \cot\theta \, d\theta = c \qquad \ldots (1)$

But $\quad \frac{r^2 + k^2}{r(r-k)(r+k)} = \frac{-1}{r} + \frac{1}{r-k} + \frac{1}{r+k}$, by partial fractions.

$$\therefore \quad \int \frac{(r^2+k^2)\,dr}{r(r^2-k^2)} = -\log r + \log(r-k) + \log(r+k)$$

ie.,
$$\int \frac{(r^2+k^2)\,dr}{r(r^2-k^2)} = \log\left(\frac{r^2-k^2}{r}\right) \qquad \ldots (2)$$

Using (2) in (1) we have,

$$\log\left(\frac{r^2-k^2}{r}\right) + \log\sin\theta = c \quad \text{or} \quad \log\left(\frac{r^2-k^2}{r}\cdot\sin\theta\right) = \log b \ \ (say)$$

Thus $\left(r - \dfrac{k^2}{r}\right)\sin\theta = b$, is the required O.T.

---

104. *Using the concept of orthogonal trajectories show that the family of curves* $r = a(\sin\theta + \cos\theta)$ *and* $r = b(\sin\theta - \cos\theta)$ *intersect each other orthogonally.*

>> We shall show that the O.T of the first family $r = a(\sin\theta + \cos\theta)$ is the other family $r = b(\sin\theta - \cos\theta)$.

Consider $r = a(\sin\theta + \cos\theta)$

$$\Rightarrow \quad \log r = \log a + \log(\sin\theta + \cos\theta)$$

Differentiating *w.r.t* θ we have,

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta}$$

Replacing $\dfrac{dr}{d\theta}$ by $-r^2\dfrac{d\theta}{dr}$ we have,

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} \quad \text{or} \quad -r\frac{d\theta}{dr} = \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$$

$$\therefore \quad \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\,d\theta = \frac{dr}{-r} \quad \text{by separating the variables.}$$

$$\Rightarrow \quad \int \frac{dr}{r} + \int \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\,d\theta = c$$

ie., $\log r - \log(\cos\theta - \sin\theta) = c$

ie., $\log\left[\dfrac{r}{\cos\theta - \sin\theta}\right] = \log k \ \ (say)$

$$\Rightarrow \quad r = k(\cos\theta - \sin\theta) \quad \text{or} \quad r = -k(\sin\theta - \cos\theta)$$

Denoting $-k = b$ we have $r = b(\sin\theta - \cos\theta)$ **as required.**

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105. *Show that the orthogonal trajectories of the family* $r(1-\cos\theta) = a$ *is the family* $r(1+\cos\theta) = b$, *where* a, b *are constants*

>> Consider $r(1-\cos\theta) = a$

$\Rightarrow$ $\log r + \log(1-\cos\theta) = \log a$

Differentiating *w.r.t* θ we have,

$$\frac{1}{r}\frac{dr}{d\theta} + \frac{\sin\theta}{1-\cos\theta} = 0$$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ and simplifying the second term we have,

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)}$$

ie., $r\frac{d\theta}{dr} = \cot(\theta/2)$ or $\tan(\theta/2)\,d\theta = \frac{dr}{r}$ by separating the variables.

$\Rightarrow$ $\int\frac{dr}{r} - \int\tan(\theta/2)\,d\theta = c$

ie., $\log r - 2\log\sec(\theta/2) = c$

ie., $\log\left[\frac{r}{\sec^2(\theta/2)}\right] = \log k$ *(say)*

$\Rightarrow$ $r\cos^2(\theta/2) = k$ or $r\cdot\frac{1}{2}(1+\cos\theta) = k$

ie., $r(1+\cos\theta) = 2k$. and let $2k = b$.

Thus $r(1+\cos\theta) = b$, **is the required O.T.**

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106. *Test for self orthogonality* $r^n = a\sin n\theta$

>> Consider $r^n = a\sin n\theta$

$\Rightarrow$ $n\log r = \log a + \log(\sin n\theta)$.

Differentiating *w.r.t* θ we have,

$$\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{n\cos n\theta}{\sin n\theta} \quad\text{or}\quad \frac{1}{r}\frac{dr}{d\theta} = \cot n\theta$$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ we have

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \cot n\theta \quad\text{or}\quad -r\frac{d\theta}{dr} = \cot n\theta$$

$\therefore \qquad \tan n\theta \, d\theta = \dfrac{dr}{-r}$ by separating the variables.

$\Rightarrow \qquad \displaystyle\int \frac{dr}{r} + \int \tan n\theta \, d\theta = c$

ie., $\quad \log r + \dfrac{\log(\sec n\theta)}{n} = c \quad$ or $\quad n \log r + \log(\sec n\theta) = nc$

ie., $\quad \log(r^n \sec n\theta) = \log b$ *(say)*

$\therefore$ **$r^n = b \cos n\theta$ is the required O.T and we conclude that the given family is not self orthogonal.**

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107. If $u(r, \theta)$ and $v(r, \theta)$ are two functions satisfying the conditions : $r \dfrac{\partial u}{\partial r} = \dfrac{\partial v}{\partial \theta}$ and $r \dfrac{\partial v}{\partial r} = \dfrac{-\partial u}{\partial \theta}$ verify that the family of curves $u(r, \theta) = a$ and $v(r, \theta) = b$ are orthogonal trajectories of each other

\>> With the background of polar curves we shall show that

$\tan \phi_1 \cdot \tan \phi_2 = -1$ where $\tan \phi = r \dfrac{d\theta}{dr}$ in general, ϕ being the angle between the radius vector and the tangent.

Consider $u(r, \theta) = a$ and differentiate *w.r.t* θ treating r as a function of θ

ie., $\quad u_r \dfrac{dr}{d\theta} + u_\theta = 0 \quad \therefore \quad \dfrac{dr}{d\theta} = \dfrac{-u_\theta}{u_r}$

We have $\tan \phi_1 = r \dfrac{d\theta}{dr}$ and hence $\tan \phi_1 = \dfrac{-r\, u_r}{u_\theta}$

Similarly for $v(r, \theta) = b$, we have $\tan \phi_2 = \dfrac{-r\, v_r}{v_\theta}$

$\therefore \qquad \tan \phi_1 \cdot \tan \phi_2 = \dfrac{(r\, u_r)(r\, v_r)}{u_\theta \cdot v_\theta}$

But $\quad r\, u_r = v_\theta$ and $r\, v_r = -u_\theta$ by data.

Hence $\tan \phi_1 \cdot \tan \phi_2 = \dfrac{(v_\theta) \cdot (-u_\theta)}{u_\theta \cdot v_\theta} = -1$

Hence the family of curves $u(r, \theta) = a$ and $v(r, \theta) = b$ are orthogonal trajectories of each other.

EXERCISES

***Find the orthogonal trajectories of the following family of curves* [1 to 4]**

1. $y = ax^2$ 2. $xy = a^2$ 3. $y = a(x+1)$

4. $x^2 + y^2 + 2\lambda y + c = 0$, λ being the parameter.

5. Show that the O.T of the family of circles having their centres on the y - axis and also passing through the origin is a family of circles passing through the origin having their centres on the x - axis.

6. Show that the family of parabolas $x^2 = 4a(y+a)$ is self orthogonal.

***Find the orthogonal trajectories of the following family of curves.* [7 to 15]**

7. $r = a \sec^2(\theta/2)$ 8. $r^2 = a^2 \sin 2\theta$ 9. $r^n = a^n \operatorname{cosec} n\theta$

10. $r = a(1 + \cos\theta)$ 11. $r^n = a^n(1 + \cos n\theta)$ 12. $r = a(1 - \sin\theta)$

13. $r = a\cos^2\theta$ 14. $r = a^\theta$

15. $\left(r - \frac{k^2}{r}\right)\sin\theta = a,$ where a is the parameter.

ANSWERS

1. $x^2 + 2y^2 = c$ 2. $x^2 - y^2 = c$

3. $x^2 + y^2 + 2x + b = 0$ 4. $x^2 + y^2 + 2\mu + c = 0$, μ is arbitrary

7. $r = b \operatorname{cosec}^2(\theta/2)$ 8. $r^2 = b\cos 2\theta$

9. $r^n = b \sec n\theta$ 10. $r = b(1 - \cos\theta)$

11. $r^n = b(1 - \cos n\theta)$ 12. $r = b(1 + \sin\theta)$

13. $r^2 = b\sin\theta$ 14. $(\log r)^2 + \theta^2 = b$

15. $\left(r + \frac{k^2}{r}\right)\cos\theta = b$